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Black Schole Model – an Econophysics Approach

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Abstract

The Black Scholes model of option pricing constitutes the cornerstone of contemporary valuation theory. However, the model presupposes the existence of several unrealistic assumptions including the lognormal distribution of stock market price processes. In the past decade or so, physicists have begun to do academic research in economics. Perhaps people are now actively involved in an emerging field often called Econophysics. Econophysics applies statistical physics methods to economical, financial, and social problems. The main goal of this study is threefold: 1) lists out the derivation of the Black-Scholes formula through the partial differential equation based on the construction of the complete "hedge portfolio", 2) to provide a brief introduction to the problem of pricing financial derivatives in continuous time; 3) and finally we will show the totality theory developed in the previous section with a concrete example.

Key Words: Econophysics, Black Scholes model, Pricing

1. INTRODUCTION

Econophysics, which is nowadays a broad interdisciplinary area, but rather as a pedagogical introduction to the mathematics (and physics?) of financial derivatives.

Econophysics concerns the use of concepts from statistical physics in the description of financial systems. Specifically, the scaling concepts used in probability theory, in critical phenomena, and in fully developed turbulent fluids. These concepts are then applied to financial time series to gain new insights into the behavior of financial markets. It is also present a new stochastic model that displays several of the statistical properties observed in empirical data.

Usually in the study of economic systems it is possible to investigate the system at different scales. But it is often impossible to write down the 'microscopic' equation for all the economic entities interacting within a given system. Statistical physics concepts such as stochastic dynamics, short- and long-range correlations, self-similarity and scaling permit an understanding of the global behavior of economic systems without first having to work out a detailed microscopic description of the same system. Econophysics will be of interest both to physicists and to economists. Physicists will find the application of statistical physics concepts to economic systems interesting and challenging, as economic systems are among the most intriguing and fascinating complex systems that might be investigated. Economists and workers in the financial world will find useful the presentation of empirical analysis methods and well formulated theoretical tools that might help describe systems composed of a huge number of interacting subsystems.

No claim of originality is made here regarding the contents of the present notes. Indeed, the basic theory of financial derivatives can now be found in numerous textbooks, written at a different mathematical levels and aiming at specific (or mixed) audiences, such as economists [1, 2, 3, 4], applied mathematicians [5, 6, 7, 8], physicists [9, 10, 11], etc.

In this paper we attempt a generalized Black Scholes formula through an econophysics. This study is threefold: 1) lists out the derivation of the Black-Scholes formula through the partial differential equation based on the construction of the complete "hedge portfolio", 2) to provide a brief introduction to the problem of pricing financial derivatives in continuous time, it contains what is the raison d'etre of the present notes; 3) and finally we will show the totality theory developed in the previous section with a concrete example. Before I finish the last Section with concludes.

2. THE BLACK SCHOLES MODEL

In order to facilitate continuity, we summarize below the original derivation of the Black Scholes model for the pricing of a European call option [1,12-15] and references therein. The European call option is defined as a financial contingent claim that enables a right to the holder thereof (but not an obligation) to buy one unit of the underlying asset at a future date (called the exercise date or maturity date) at a price (called the exercise price). Hence, the option contract, has a payoff of $\max(S_T - E,0) = (S_T - E)^+$ on the maturity date where S_T is the stock price on the maturity date and E is the exercise price.

We consider a non-dividend paying stock, the price process of which follows the geometric Brownian motion with drift $S_t = e^{(\mu t+\sigma W_t)}$. The logarithm of the stock price $Y_t = In\ S_t$ follows the stochastic differential equation

$$dY_t = \mu dt + \sigma dW_t \qquad (1)$$

where W_t is a regular Brownian motion representing Gaussian white noise with zero mean and δ correlation in time i.e. $E(dW_t dW_{t'}) = dtdt'\delta(t-t')$ on some filtered probability space $(\Omega,(F_t),P)$ and μ and σ are constants representing the long term drift and the noisiness (diffusion) respectively in the stock price.

Application of Ito's formula yields the following SDE for the stock price process

$$dS_t = \left(\mu + \frac{1}{2}\sigma^2\right)S_t dt + \sigma S_t dW_t \qquad (2)$$

Let $C(S,t)$ denote the instantaneous price of a call option with exercise price E at any time t before maturity when the price per unit of the underlying is S. It is assumed that $C(S,t)$ does not depend on the past price history of the underlying. Applying the Ito formula to $C(S,t)$ yields

$$dC = \left(\mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}\right)dt + \frac{\partial C}{\partial S}\sigma S dW, \qquad (3)$$

The original option-pricing model propounded by Fischer Black and Myron Scholes envisaged the construction of a "hedge portfolio", Π, consisting of the call option and a short sale of the underlying such that the randomness in one cancels out that in the other. For this purpose, we make use of a call option together with $\partial C/\partial S$ units of the underlying stock.

We then have, on applying Ito's formula to the "hedge portfolio", Π,:-

$$\frac{d\Pi}{dt}=\frac{d}{dt}\left[C(S,t)-S\frac{\partial C(S,t)}{\partial S}\right]=\frac{dC(S,t)}{dt}-\frac{\partial C}{\partial S}\cdot\frac{dS}{dt} \qquad \textbf{(4)}$$

where the term involving $\frac{d}{dt}\left(\frac{\partial C}{\partial S}\right)$ has been assumed zero since it envisages a change in the portfolio composition. On substituting from eqs. (2) & (3) in (4), we obtain

$$\frac{d\Pi}{dt}=\frac{dC(S,t)}{dt}-\left(\mu+\frac{1}{2}\sigma^2\right)S\frac{\partial C(S,t)}{\partial S}-\sigma S\frac{\partial C}{\partial S}\frac{dW}{dt}=\frac{\partial C(S,t)}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 C(S,t)}{\partial S^2} \qquad \textbf{(5)}$$

We note, here, that the randomness in the value of the call price emanating from the stochastic term in the stock price process has been eliminated completely by choosing the portfolio $\Pi=C(S,t)-S\frac{\partial C(S,t)}{\partial S}$. Hence, the portfolio Π is free from any stochastic noise and the consequential risk attributed to the stock price process.

Now $\frac{d\Pi}{dt}$ is nothing but the rate of change of the price of the so-called riskless bond portfolio i.e. the return on the riskless bond portfolio (since the equity related risk is assumed to be eliminated by construction, as explained above) and must, therefore, equal the short-term interest rate r i.e.

$$\frac{d\Pi}{dt}=r\Pi. \qquad (6)$$

In the original Black Scholes model, this interest rate was assumed as the risk free interest rate r, further, assumed to be constant, leading to the following partial differential equation for the call price:-

$$\frac{d\Pi}{dt} = r\Pi = r\left[C(S,t) - S\frac{\partial C(S,t)}{\partial S}\right] = \frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2}$$

or equivalently

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S \frac{\partial^2 C(S,t)}{\partial S^2} + rS\frac{\partial C(S,t)}{\partial S} - rC(S,t) = 0 \qquad (7)$$

which is the famous Black Scholes PDE for option pricing with the solution:-

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2) \qquad (8)$$

where

$$d_1 = \frac{\log\left(\frac{S}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\log\left(\frac{S}{E}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and}$$

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}} dx$$

3. THE STANDARD MODEL OF FINANCE

3.1 Portfolio dynamics and arbitrage

Consider a financial market with only two assets: a risk-free bank account B and a stock S. In vector notation, we write $\vec{S}(t) = (B(t), S(t))$ for the asset price vector at time t. A portfolio in this market consists of having an amount x_0 in the bank and owing x_1 stocks.

The vector $\vec{x}(t)=(x_0(t),x_1(t))$ thus describes the time evolution of your portfolio in the (B,S) space. Note that $x_i<0$ means a short position on the *ith* assets, i.e., you 'owe the market' $|x_i|$ units of the *ith* asset.

Let us denote by $V_{\vec{x}}(t)$ the money value of the portfolio $\vec{x}(t)$:

$$V_{\vec{x}}(\vec{S})=x_0B+x_1S, \qquad (9)$$

Where the time dependence has been omitted for clarity. We shell also often suppress the subscript from $V_{\vec{x}}(t)$ when there is no risk of confusion about to which portfolio we are referring.

A portfolio is called self-financing if no money is taken from it for 'consumption' and no additional money is invested in it, so that any change in the portfolio value comes solely from changes in the assets prices. More precisely, a portfolio $\vec{x}$ is self-financing if its dynamics is given by

$$dV_{\vec{x}}(t)=\vec{x}(t)d\vec{S}(t), \quad t\geq 0. \qquad (10)$$

The reason for this definition is that in the discrete-time case, i.e., $t=t_n, n=0,1,2...$, the increase in wealth, $\Delta V(t_n)=V(t_{n+1})-V(t_n)$, of a self-financing portfolio over the time interval $t_{n+1}-t_n$ is given by

$$\Delta V(t_n)=\vec{x}(t_n).\Delta\vec{S}(t_n), \qquad (11)$$

where $\Delta\vec{S}(t_n)\equiv\vec{S}(t_{n+1})-\vec{S}(t_n)$. This means that over the time interval $t_{n+1}-t_n$ the value of the portfolio varies only owing to the changes in the assets prices themselves, and then at time t_{n+1} re-allocate the assets within the portfolio for the next time period. Equation (10) generalized this idea for the continuous-time limit. If furthermore we decide on the makeup of the portfolio by looking only at the current prices and not on past times, i.e., if

$\vec{x}(t)=\vec{x}(t,S(t))$,

then the portfolio is said to be Markovin.

3.2 The Black-Schools model for option pricing

The two main assumption of the Black-Scholes model are:

I. There are two assets in the market, a bank account B and a stock S, whose price dynamics are governed by the following differential equations

$$dB = rBdt, \tag{12}$$

$$dS = \mu Sdt + \sigma SdW, \tag{13}$$

Where r is the risk-free interest rate, $\mu > 0$ is the stock mean rate of return, $\sigma > 0$ is the volatility, and $W(t)$ is the standard Brownian motion or Wiener process.

II. The market is free of arbitrage.

Besides these two crucial hypotheses, there are additional simplifying (technical) assumptions, such as: (iii) there is a liquid market for the underlying asset S as well as for the derivative one wishes to price, (iv) there are no transaction costs (i.e., no bid-ask spread), and (v) unlimited short selling is allowed for an unlimited period of time. It is implied by (12) that there is no interest-rate spread either, that is, money is borrowed and lent at the same rate r. Equation (13) also implies that the stock pays no dividend. This last assumption can be relaxed to allow for dividend payments at a known (i.e., deterministic) rate.

We shall next describe how derivatives can be “rationally’ priced in the Black-Scholes model. We consider first a European call option for which a closed formula can be found. Let us then denoted by $C(S,t;K,T)$ the present value of a European call option with strike price K and expiration date T on the underlying stock S. For ease of notation we shell drop the parameters K and T and simply $C(S,t)$. For later use, we note here that according to Ito formula, with $a = \mu S$ and $b = \sigma S$, the option price C obeys the following dynamics

$$dC = \left[\frac{\partial C}{\partial t} + \mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right]dt + \sigma S\frac{\partial C}{\partial S}dW. \tag{14}$$

In what follows, we will arrive at a partial differential equation, so called Black-Scholes Equation (BSE), for the option price $C(S,t)$. For pedagogical reasons, we will present two alternative derivatives of the BSE using two distinct but arguments:

The Δ-Hedging Portfolio and
The Replicating Portfolio

The Δ-Hedging Portfolio

In the binomial model the self financing Δ-hedging portfolio, we the self-financing Δ-hedging portfolio, consisting of a long position on the option and a short position on Δ stocks. The value $\Pi(t)$ of this portfolio is

$$\Pi(t) = C(S,t) - \Delta S$$

Since the portfolio is self-financing, it follows from (10) that Π obeys the following dynamics

$$d\Pi = dC - \Delta dS, \tag{15}$$

which in (13) and (14) becomes

$$d\Pi = \left[\frac{\partial C}{\partial t} + \mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - \mu\Delta S\right]dt + \sigma S\left(\frac{\partial C}{\partial S} - \Delta\right)dW. \tag{16}$$

We can now eliminate the risk (i.e., the stochastic term containing dW) from this portfolio by choosing

$$\Delta = \frac{\partial C}{\partial S} \tag{17}$$

Inserting this back into (16), we then find

$$d\Pi = \left[\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right]dt \tag{18}$$

Since we now have a risk-free (i.e., purely deterministic) portfolio, it must yield the same rate of return as the bank account, which means that

$$d\Pi = r\Pi dt. \tag{19}$$

Comparing (18) with (19) and using (15) and (17), we then obtain the Black Scholes Equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} rS\frac{\partial C}{\partial S} - rC = 0, \tag{20}$$

which must be solved to the following boundary condition

$$C(S,T) = \max(S-K,0). \tag{21}$$

The solution to the above boundary-value problem can be found explicitly, but before going into that it is instructive to consider an alternative derivation of the BSE.

The Replicating Portfolio

Here we will show that it is possible to form a portfolio on the (B,S) market that replicates the option $C(S,t)$, and in the process of doing so we will arrive at the BSE. Suppose then that there is indeed a self-financing portfolio $\vec{x}(t) = (x(t), y(t))$, whose value $Z(t)$ equals the option price $C(S,t)$ for all time $t \le T$:

$$Z = xB + yS = C\ , \tag{22}$$

where we have omitted the time-dependence for brevity. Since the portfolio is self-financing it follows that

$$dZ = xdB + ydS = (rxB + \mu yS)dt + \sigma ySdW\ . \tag{23}$$

But by assumption we have $Z = C$ and so $dZ = dC$. Comparing (23) with (14) and equating the coefficients separately in both dW and dt, we obtain

$$y = \frac{\partial C}{\partial S}, \tag{24}$$

$$\frac{\partial C}{\partial t} - rxB + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0. \tag{25}$$

Now from (22) and (24) we get that

$$x = \frac{1}{B}\left[C - S\frac{\partial C}{\partial S}\right], \tag{26}$$

which inserted into (25) yields again the BSE (20), as the reader can easily verify.

We have thus proven, by direct construction, that the option C can we be replicated in the (B,S)-market by the portfolio (x,y), where x and y are given in (26) and (24), respectively, with option price C being the solution of the BSE (with the corresponding boundary condition).

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4. ENTIRETY IN THE BLACK – SCHOLES MODEL

We have seen above that it is possible to replicate a European call option $C(S,t)$ using an appropriate self-financing portfolio in the (B,S) market. Looking back at the arguments given in sec 3.4, we see that we never actually made use of the fact that the derivative in question was a call option – the nature of the derivative appeared only through the boundary condition (21). Thus, the derivation of the BSE presented there must hold for any contingent claim!

To state this fact more precisely, let $F(S,t)$ represent the price of an arbitrary European contingent claim with payoff $F(S,T)=\Phi(S)$, where Φ is a known function. Retracing the steps outlined in sec 3.4, we immediately conclude that the price $F(S,t)$ will be the solution to the following boundary-value problem

$$\frac{\partial F}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2}+rS\frac{\partial F}{\partial S}-rF=0, \tag{27}$$

$$F(S,T)=\Phi(S) \tag{28}$$

Furthermore, if we repeat the arguments of preceding sub section and transform the Black-Scholes equation (27) into the heat equation, we obtain that $F(S,t)$ will be given by

$$F(S,t)=\frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty}\Phi(x')e^{-(x-x')^2/4\tau dx'}, \tag{29}$$

where $\Phi(x)$ denote the payoff function in terms of the dimensionless variable x. expressing this result in terms of the original variables S and t yields a generalized Black-Scholes formula

$$F(S,t)=\frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\int_{0}^{\infty}\Phi(S')e^{\left[\ln\left(\frac{S'}{S}\right)-\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right]^2}\frac{dS'}{S'}. \tag{30}$$

In summary, we have shown above that the Black Scholes model is complete. A market is said to be complete if every contingent claim can be replicated with a self-financing portfolio on the primary assets. Our 'proof of Entirety' given above is, of course, valid only for the case of European contingent claim with a simply payoff function $\Phi(S)$; it does not cover, for instance, path dependent derivatives. It is possible however to give a formal proof that arbitrage-free models, such as the Black Scholes model, are indeed complete.

Comparing the generalized Black-Scholes formula (30) with the pdf of the geometric Brownian motion given

$$p(S,t,S_0,t_0)=\frac{1}{\sqrt{2\sigma^2\tau S}}\exp\left\{\frac{\left[In\left(\frac{S}{S_0}\right)-\left(\mu-\frac{1}{2}\sigma^2\right)\tau\right]^2}{2\sigma^2\tau}\right\} \quad (31)$$

the Geometric Brownian motion is the basic model for stock price dynamics in the Black-Scholes framework.

We see that the former can be written in a convenient way as

$$F(S,t)=e^{-r(T-t)}E_{t,S}^{Q}\left[\Phi(S_T)\right], \quad (32)$$

where $E_{t,S}^{Q}[\cdot]$ denotes expectation value with respect to the probability density of Geometric Brownian formula $\mu=r$, initial time t, final time T, and initial value S;

In other words, the present value of a contingent claim can be computed simply as its discounted expected value at maturity, under an appropriate probability measure.

5. CONCLUSION

In these notes, I tried to present a basic introduction to an interdisciplinary area that has become known, at least among physicists working on the field, as Econophysics. I started out by giving the two fundamental derivatives of Black-Scholes model for pricing financial derivatives. After this motivation, I offered to introduction to the problem of pricing financial derivatives in continuous time with standard model of finance, namely, the Black-Scholes model for pricing financial derivatives. Finally, I briefly reviewed the totality theory developed in the previous section with concrete example. Some recent work done mostly, but not exclusively, by physicists that have produced evidences that the Standard Model of Finance (SMF) may not fully describe real markets. In this context, some possible extensions of the Black-Scholes model were considered.

I should like to conclude by mentioning that other alternatives approaches to the problem of pricing financial derivatives have been proposed by physicists, using methods originally developed to treat physical problems. For instance, the option pricing problem was recently discussed in the context of the so-called non-extensive statistical mechanics [21]. A "Hamiltonian formulation" for this problem was also given in which the resulting "generalized Black- Scholes" equation is formally solved in terms of path integrals [22].

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