Do Price Limits Change Investors Behavior: Evidence From the Tunisian Stock Exchange

Eymen Errais (Corresponding author)

LAREQUAD & Tunis Business School

University of Tunis, Tunisia

E-mail: eymen@stanfordalumni.org

Jawhar Albacha

Department of Quantitative Methods

Faculty of Economic and Managerial Sciences

University of Tunis El Manar, Tunis, Tunisia

Received: June 19, 2019 Accepted: August 19, 2019 Published: August 26, 2019

doi:10.5296/ijafr.v9i3.15029 URL: https://doi.org/10.5296/ijafr.v9i3.15029

Abstract

Several stock exchanges are subject to some authorities' regulations, constraints and limitations such as price limits. In this paper, we address the issue of microstructure effects due to price limits. In particular, stocks returns show a non-normal behavior in the case of price limits. This non-normality could be coming from a change in investors' behavior, **valuation effect,** or a change in the stock returns statistical properties, **value effect**. In order to analyze this joint-hypothesis, we defined a new normality test that takes into account the truncation effect, the Truncated Jacobi-Bellman (TJB) test. Our results show that the Value effect has a limited explanation of the stock return behavioral change.

Keywords: Price limits, Truncated time series, Truncated normal distribution, JB test, Maximum likelihood estimator

1. Introduction

To limit asset prices volatility, many stock exchanges impose daily price limits, i.e. today's prices trade within a certain range of yesterday's closing prices. There is a long-haul debate surrounding the usefulness of price limits which dragged the financial community for decades. On one hand, *Price Limit Advocates* claim that price limits decrease stock price volatility, counter overreaction, and does not interfere with trading activity. On the other hand, *Price Limit Critics* advance that price limits cause negative effects such as higher volatility levels on subsequent days (volatility spillover hypothesis), preventing prices from efficiently reaching their equilibrium level (delayed price discovery hypothesis), and interfering with trading due to limitations imposed by price limits (trading interference hypothesis).

Many academics demonstrated an interest to study several phenomena related to price limits. Kim and Rhee (1997) stated that price limits prevent stock prices from falling below or rising above predetermined boundaries. This enables them to control a potential volatility by establishing constraints and providing time for rational reassessment during panic times. They check whether price limits increase volatility levels on the upcoming days, if they do not allow prices to reach their equilibrium levels and whether they interfere with trading because of the limitations they impose. Their study is conducted on the Tokyo stock exchange price limits system. The authors used daily stock price between 1989 and 1992 and then compared stocks reaching price limits to those that almost attained their limits on volatility, price continuation, and reversal and trading activities levels. The findings have suggested that volatility does not return to a normal level as fast as those who did not attain the price limit. In addition, price continuation occurs more frequently when the price limits are reached. Finally, it is documented that price limits cause trading activities to increase. These results had the literature questioning the effectiveness of price limits in reducing volatility.

Chen et al (2005) checked the effectiveness of the price limits system in the Shanghai and the Shenzhen stock exchanges, more specially the A shares, by testing the volatility spillover hypothesis of Fama (1989), the delayed price discovery hypothesis of Fama (1989) and the trading interference hypothesis of Lauterbach and Ben-Zion (1993). Moreover, a test was conducted to determine if stocks, that hit the price limits, have certain attributes such as being volatile, actively trading, or having small capitalization, etc. To do so, they used Chinese a share individual stock prices and volume between December 1996 and December 2003. The findings indicated that the effect of price limits is asymmetric for the upwards and downwards movement and different for the bullish and bearish sample periods. During the bullish periods, price limits significantly reduce the stock volatility for the downwards movement mainly. It is the opposite case for the bearish sample period. Moreover, actively traded stock hits the price limits more frequently, especially the lower boundary when the market is bearish while stock with high book to market hit the upper limit more often. Lastly, a lack of evidence indicates that price limits interfere negatively with the trading process.

Wang et al (2014) studied the effects of price limits on the Chinese stock market during periods of global instability. The purpose was to examine the characteristics of stocks that hit the price limits more frequently during periods of turmoil. Specially, these periods are the Asian financial crisis of 1997 and the global financial crisis of 2008. To do so, they tackled volatility spillovers, delayed price discovery and trading interference with such periods using daily A-share stock prices and trading volume from the Shanghai stock exchange and the

Shenzhen stock exchange. The findings are quite interesting. First, the price limits mechanism increased the volatility significantly, especially in the downward movement, during the global financial crisis. Secondly, they found that the efficient price discovery was delayed by price limits. Moreover, throughout the upward movement, price limits interfere in the trading activity one day after the stock hits the limit. Lastly, actively traded stocks in the property and industrial sectors which are highly and positively correlated with the market reach the price limits frequently.

Errais and Bahri (2016) claim that for investors trading across assets and countries with different price limits, volatility estimates (measured by standard deviation) are biased. This biasness is because price limits are imposed. In situations like these, equilibrium prices are unobservable and observed prices are truncated. Their methodology consists in using Censored Stochastic Volatility model (CSV) and optional pricing. The findings show that the returns of stocks traded on markets with price limits exhibit an option lookalike payoff. Consequently, when options are inexistent in a market, traders gain options payoff as well as the regular linear payoff observed in stocks.

Mathematically, if we denote today's price by $\{p_t^0\}$ we can write:

$$p_t^0 \in [d_t p_{t-1}^0, u_t p_{t-1}^0] \qquad d_t < u_t \tag{1}$$

By simply dividing $p^0{}_t$ by $p^0{}_{t-1}$ from (1) we can notice that the returns are limited as well:

$$R_t^0 := \frac{p_t^0}{p_{t-1}^0} \in [d_t, u_t] \tag{2}$$

If we suppose that these limits are symmetric and time independent, there exists a limit $0 \leq l \leq 1$ such as:

$$d_t = 1 - l \, , \, u_t = 1 + l$$

With this configuration, one can think about censored time series to model $\{R_t^0\}$ as R_t^0 being a stochastic process with a finite support. We will see in this paper that the observed returns $\{R_t^0\}$ have similar but not identical features to truncated time series when we establish its link with the shadowing returns. Thus the price limits don't boil down to truncating or censoring a time series.

The purpose of this paper is to address the following questions: How does limited price rules influence both traders' decision making and valuation of assets? How can we model the observed returns to link them with equilibrium prices? To do so, this paper first analyzes the impact of price limits requirements on the behavior of stock returns and on decision parameters estimation (mean, variance, covariance etc...). Second, it proposes an estimation methodology to reconstruct the unobserved returns density function.

The remainder of the paper is organized as follows: Section two will discuss the impact of price limitations on the stock return behavior. Section 3 sets up the theoretical background of censored and truncated time series. Section 4 will present an estimation methodology of shadowing returns density function based on maximum entropy methods and kernel estimation. Section 5 analyzes the relevant normality tests for truncated times series. Section 6 runs empirical examples from the Tunisian Stock Exchange. Section 7 concludes.

2. Impact of Price Limits on the Behavior of Stock Returns

The impacts of price limits on the behavior of stock returns can be divided into two major effects:

Valuation effect: imposing price limits on future returns influences the present value of the asset. If we suppose that there is a stochastic discount factors series $\{m_t\}$, the unconditional equilibrium prices $\{\pi_t\}$ and the conditional equilibrium prices $\{p_t\}$ can be defined as:

$$\pi_t := \mathbb{E}_t\left[\sum_{i\in\{x|t+x\in T\}} m_{t+i} X_{t+i}\right] \tag{3}$$

$$p_t := \mathbb{E}_t\left[\sum_{i\in\{x|t+x\in T\}} m_{t+i} \bar{X}_{t+i}\right] \tag{4}$$

with:

$\{X_t\}$: Cash flow series generated by the asset.

$\{\bar{X}_t\}$: Cash flow series generated by the asset when price limits are imposed.

Value effect: Equation (4) does not guarantee that p_t is in the price limit range. If we suppose that p_t at time *t* is known, we can define the observed price $\{p_t^0\}$ as follows:

$$p_t^0 := \arg\min_{p\in\mathcal{A}_t} |p_t - p| \qquad \mathcal{A}_t := [d_t p_{t-1}^0, u_t p_{t-1}^0] \tag{5}$$

Equation (5) means that the observed price is the most accepted price close to the *shadowing price*. This is true if we suppose that the investor will execute the price p_t. With minor manipulations we can reformulate (5) to get a more explicit relationship between p_t^0 and p_t:

$$p_t^0 = \begin{cases} d_t p_{t-1}^0 & \text{if} \dfrac{p_t}{p_{t-1}^0} < d_t \\ p_t & \text{ifd}_t \leqslant \dfrac{p_t}{p_{t-1}^0} \leqslant u_t \\ u_t p_{t-1}^0 & \text{if} \dfrac{p_t}{p_{t-1}^0} > u_t \end{cases} \tag{6}$$

From (6), we can define the observed return as:

$$R_t^0 := \frac{p_t^0}{p_{t-1}^0} = \begin{cases} d_t & \text{if } \frac{p_t}{p_{t-1}^0} < d_t \\ \frac{p_t}{p_{t-1}^0} & \text{if} d_t \leqslant \frac{p_t}{p_{t-1}^0} \leqslant u_t \\ u_t & \text{if } \frac{p_t}{p_{t-1}^0} > u_t \end{cases} \tag{7}$$

If we have $d_t \leq \frac{p_t}{p_{t-1}^0} \leq u_t$, we cannot guarantee that $R_t^0 = R_t$ as p_{t-1}^0 can be different from p_{t-1}.

To make the treatment simpler, we will suppose that price limits are time independent and symmetric, thus, we will have $d_t = 1 - l$ and $u_t = 1 + l$. The following example will help illustrating the difference between a simple censored series and the process described by (7)

Example 1.

Suppose the following configuration:

$$p_{t-1} = (1 + l + d)\, p_{t-2} \text{ with } d \geqslant 0$$

$$p_t = (1 + r_t)\, p_{t-1} \text{ with } 0 \leqslant r_t < l$$

$$p_{t-2}^0 = p_{t-2}$$

According to (6):

$$\frac{p_{t-1}}{p_{t-2}^0} = (1 + l + d) \Rightarrow R_{t-1}^0 = 1 + l$$

$$\frac{p_t}{p_{t-1}^0} = \frac{(1 + r_t)(1 + l + d)}{(1 + l)} = (1 + r_t) + \frac{d\,(1 + r_t)}{(1 + l)}$$

For

$$d \geqslant (l - r_t)\, \tfrac{1 + l}{1 + r_t};\ R_t^0 = 1 + l$$

for

$$d < (l - r_t)\, \tfrac{1 + l}{1 + r_t};\ p_t^0 = p_t \text{ and } R_t^0 = (1 + r_t) + \tfrac{d\,(1 + r_t)}{(1 + l)}$$

From the example above we can conclude that

$$\mathbb{P}\left[R_t^0 = 1 + l\right] \neq \mathbb{P}\left[R_t \geqslant 1 + l\right]$$

$$\mathbb{P}[a < R_t^0 < b] \neq \mathbb{P}[a < R_t < b] \quad ; \quad a, b \in [1 - l; 1 + l]$$

Proposition 1.

If $R_{t-1}^0 \neq (1 \pm l)$ Then

$$R_t^0 = \begin{cases} 1-l & \text{if} R_t < 1-l \\ R_t & \text{if } 1-l \leqslant R_t \leqslant 1+l \\ 1+l & \text{if} R_t > 1+l \end{cases}$$

Proof: If $\boldsymbol{R_{t-1}^0 \neq (1 \pm l)}$ then $R_{t-1}^0 = \frac{p_{t-1}}{p_{t-2}^0} \Leftrightarrow p_{t-1} = R_{t-1}^0 . p_{t-2}^0 = p_{t-1}^0$. Thus, using 7 we have:

$$R_t^0 = \begin{cases} 1-l & \text{if} R_t < 1-l \\ \frac{p_t}{p_{t-1}^0} = \frac{p_t}{p_{t-1}} = R_t & \text{if } 1-l \leqslant R_t \leqslant 1+l \\ 1+l & \text{if} R_t > 1+l \end{cases}$$

Proposition (2) allows us to make a link between the process of observed returns $\{R_t\}$ and censored and truncated variables.

3. Truncation

Truncation and censoring are sampling-related phenomena. Truncation occurs when the sample is drowning from a non-fully representative subpopulation, while censoring occurs when a group of values is replaced by a unique value. Greene (2003) used the example of studies of income based on incomes above or below some poverty line to explain the difference between the two phenomena: Truncation occurs when we totally neglect observations that are out of the subpopulation studied i.e. observations with income higher or lower than the chosen poverty line while censoring occurs when we replace these observations by higher than/lower than the chosen poverty line.

3.1 A Brief History

The early statistical treatment of truncation dates to 1897. The year in which Galton (1997) published his *An Examination into The Registered Speeds of American Trotting Horses*, with remarks on their value as hereditary data. In the article, the author studied the speed of trotting horses from samples published by The American Trotting Association. However, the association did not report the speed of unsuccessful trotters, that is, the horses with a speed below a certain threshold. This is what is called in modern econometrics' language a left truncation. Galton (1997) considered that his sample is normal$\mathcal{N}(\mu, \sigma)$ and used the sample mode as an estimator for μ. Then he used sample inter-quartile to estimate the standard deviation. This procedure was judged satisfactory of the author's needs.

Pearson (1902) criticized this procedure and proposed the use of fitting parabolas to the logarithms of the sample frequencies, however, the results were slightly different from those found by Galton (1997). Later, Pearson continued his investigation on the truncated normal samples. Pearson and Lee (1908) used the method of moments to derive estimators of the mean and standard deviation of left truncated normal distribution. Fisher (1931) used the maximum likelihood method to estimate the normal distribution parameters.

The treatment of censored samples was on standby until 1937. Bliss and Stevens (1937) derived maximum likelihood equations to estimate normal parameters for singly and doubly truncated.

Further development of statistical methods on how to deal with truncated and censored samples were continued with many other scholars such as Cohen, Saw, Whitten, etc...

3.2 Parallel Truncation

This subsection will focus on parallel truncation that is when we truncate the distribution from above and below:

$$X \mid d \leqslant X \leqslant u \qquad d, u \in \mathbb{R}$$

3.2.1 Effects of Truncation on Distribution Moments and Characteristics

The study of truncated moments may be of negligible importance in inferring the full population moments if we have no or limited information on the population distribution. In fact, the mean of any continuous random variable on a finite support is finite even if this random variable is the truncation of a random variable that has an infinite or even undefined mean. To spot this, the Cauchy distribution can be used because it has an undefined mean. There are two main reasons for exploring some properties of the truncated Cauchy distribution: the first reason is that the Cauchy distribution is amongst the few stable distributions that has a density function. The second reason is that the presence of fat tails in returns distribution (Lux (1998), Tsay (2005), etc...) which is captured with the Cauchy distribution.

Example 2.

Let $X \sim Cauchy(x_0, \alpha)$, We have

$$f_{X \mid d \leqslant X \leqslant u}(x) := \frac{f_X(x)}{\mathbb{P}(d \leqslant X \leqslant u)} = c.\frac{1}{\left[1 + \left(\frac{x - x_0}{\alpha}\right)^2\right]} \qquad d \leqslant x \leqslant u$$

Where

$$c := \left[\alpha\left(\arctan\left(\frac{u - x_0}{\alpha}\right) - \arctan\left(\frac{d - x_0}{\alpha}\right)\right)\right]^{-1}$$

Now we can calculate for example the mean of the truncated variable:

$$\mathbb{E}(X \mid d \leqslant X \leqslant u) = \int_d^u x.\, f_{X \mid d \leqslant X \leqslant u}(x)\, dx \quad = \frac{c\,\alpha^2}{2}\left[\ln\left((x - x_0)^2 + \alpha^2 + x_0\,\alpha \arctan\left(\frac{x - x_0}{\alpha}\right)\right)\right]_d^u \in \mathrm{R}$$

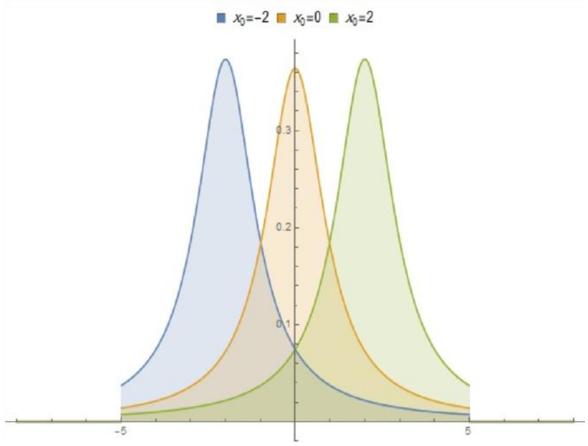


Figure 1. (-5, 5)-Truncated $Cauchy(x_0, 1)$ PDF

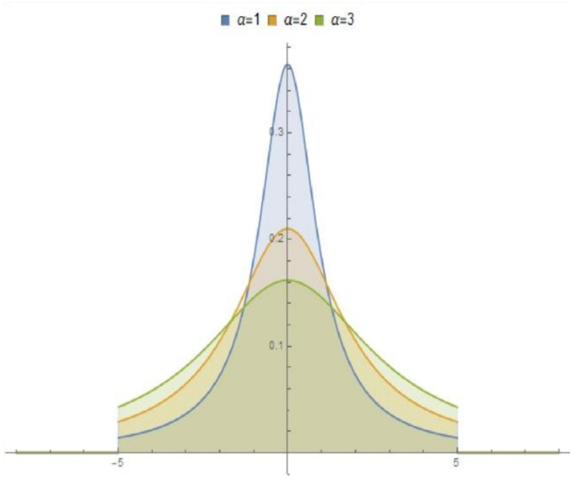


Figure 2. (-5, 5)-Truncated $Cauchy(0, \alpha)$ PDF

However, having a prior knowledge of the full population distribution may lead to a complete knowledge of its moments from the truncated ones. For instance, this is the cause for normal random variables.

Example 3.

Let $X \sim \mathcal{N}(\mu, \sigma)$, We have

$$f_{X|d \leqslant X \leqslant u}(x) := \frac{f_X(x)}{\mathbb{P}(d \leqslant X \leqslant u)} = \frac{1}{\sigma} \frac{\phi(\frac{x-\mu}{\sigma})}{\Phi(\bar{u}) - \Phi(\bar{d})} \quad d \leqslant x \leqslant u \tag{8}$$

Where $\bar{u} := \frac{u-\mu}{\sigma}$, $\bar{d} := \frac{d-\mu}{\sigma}$, $\varphi()$: the standard normal density function and $\Phi()$ The standard normal cumulative density function

Similarly to the untruncated normal distribution, we can relate the truncated moments with a recursive formula (Orjebin, 2014):

International Journal of Accounting and Financial Reporting
ISSN 2162-3082
2019, Vol. 9, No. 3

$$m_k = (k-1)\,\sigma^2\, m_{k-2} + \mu\, m_{k-1} - \sigma\, \frac{b^{k-1}\,\phi(\upsilon) - a^{k-1}\,\phi(\delta)}{\Phi(\upsilon) - \Phi(\delta)} \tag{9}$$

With

$$m_k := \mathbb{E}(X\,|\,d \leqslant X \leqslant u)\,\mathrm{for} k \geqslant 1,\, m_0 = 1 \,\mathrm{and}\, m_{-1} = 0$$

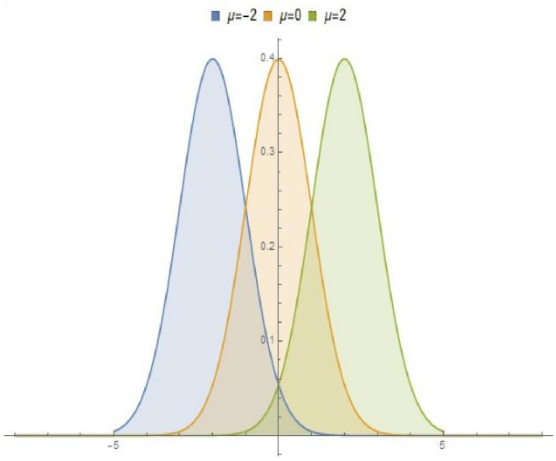


Figure 3. (-5, 5)-Truncated $\mathcal{N}(\mu, 1)$ PDF

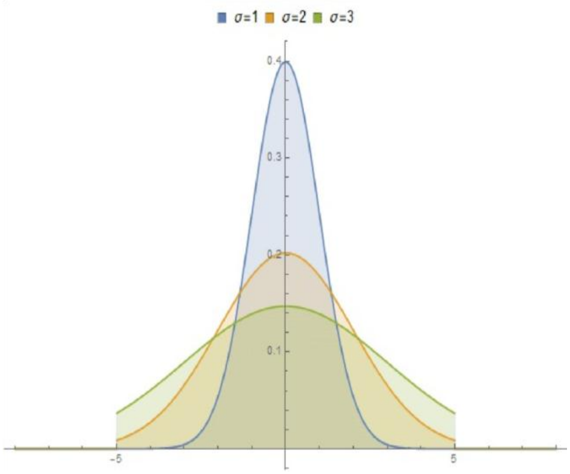


Figure 4. (-5, 5)-Truncated $\mathcal{N}(0, \sigma)$ PDF

4. Estimation Approach

In this section, we will develop parametric method to estimate a reconstruction of the parent distribution of returns. We will limit our focus on the special case of log normal returns. Based on the obtained results in section 2, some restrictions of the data should be imposed in order to obtain a truncated or censored sample. However, this article will focus on using the truncation method and leave the case of censoring for further investigation.

From the results obtained in section, we should proceed to some restrictions of the data in order to obtain a truncated or censored sample. However, in this article we will focus only on using truncation method and leave the case of censoring for further investigations. In this section we will restrict the full sample T to τ define as follows:

$$\tau := \{t \in T \,|\, R_t \neq (1 \pm l) \,\text{and}\, \mathrm{R}_{t-1} \neq (1 \pm l)\} \,\text{with}\,\text{card}(\tau) = \theta$$

We can easily see that in the restricted sample *the* observed returns coincide with shadowing returns. It's thus a truncated sample of the shadowing price.

The general hypothesis retained here is that $\{R_t\}$ is an iid sample of the variable $R|u < R < d$. statistically speaking, the object of this section is to estimate the distribution of R.

The log-normality of returns is an assumption that is commonly used in financial literature (Tsay, 2005). It combines two important features:

- It has a bijective link with normal distribution : This makes the switch between the two distributions quite easy and direct
- It can model the boundedness from below of gross returns

The first step is to log-transform the observed return to obtain $\{r_t\}$ which is now an i.i.d sample of $r|\delta < r < v$ where $r \coloneqq \log(R)\,; v \coloneqq \log(u)\,; \delta \coloneqq \log(d)$.

As *R* is considered here to follow a log normal distribution, *r* will follow a normal distribution with mean *and* standard deviationσ. In order to estimate the population parameters, we will only need the first and second order moments estimators. However, we will need the third and fourth moments estimators in order to test the hypothesis of log-normality.

From 8, we have

$$r \sim \mathcal{N}(\mu, \sigma)$$

$$f_{r|\delta<r<v}(x) = \frac{1}{\sigma}\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi(\bar{v}) - \Phi(\bar{\delta})} \quad d \leqslant x \leqslant u$$

Cohen (1950) and Cohen (1959) used two ways for estimating the moments of normal distribution from truncated samples using the method of maximum likelihood then the method of moments.

4.1 The MLE Estimator

By using the results found in (8) we can deduce directly the likelihood function:

$$\mathcal{L}(\{r_t^0\}_{t\in\tau}; \mu; \sigma) := \prod_{t\in\tau} \frac{1}{\sigma}\frac{\phi\left(\frac{r_t-\mu}{\sigma}\right)}{\Phi(\bar{v}) - \Phi(\bar{\delta})}$$

By applying the log we can find:

$$\log\mathcal{L}=\sum_{t\in\tau}\left[-(\log\sigma+\log(\Phi(\bar{\upsilon})-\Phi(\bar{\delta})))+\log\left(\phi\left(\frac{r_t-\mu}{\sigma}\right)\right)\right]$$
$$=\sum_{t\in\tau}\left[-(\log\sigma+\log(\Phi(\bar{\upsilon})-\Phi(\bar{\delta})))-\frac{1}{2}\left(\frac{r_t-\mu}{\sigma}\right)^2+\log\left(\frac{1}{\sqrt{2\pi}}\right)\right]$$

As we can see, log-likelihood is more complex than its equivalent in the case of no truncation. This is due to the presence of the term $\log(\Phi(\bar{\upsilon})\text{-}\,\Phi(\bar{\delta}))$ which incorporates μ and σ. By taking the derivative we obtain:

$$\nabla_{\mu,\sigma}\log\mathcal{L}=\begin{bmatrix}\frac{T}{\sigma}\frac{\varphi(\bar{\upsilon})-\varphi(\bar{\delta})}{(\Phi(\bar{\upsilon})-\Phi(\bar{\delta}))}+\frac{1}{\sigma^2}\sum_{t\in\tau}(r_t-\mu)\\ -\frac{T}{\sigma^2}\frac{\bar{\upsilon}\varphi(\bar{\upsilon})-\bar{\delta}\varphi(\bar{\delta})}{(\Phi(\bar{\upsilon})-\Phi(\bar{\delta}))}-\frac{T}{\sigma}+\frac{1}{\sigma^3}\sum_{t\in\tau}(r_t-\mu)^2\end{bmatrix}'$$

The maximum likelihood estimator of μ and σ, denoted $\hat{\mu}$ and $\hat{\sigma}$, can be obtained by equating the gradient to 0.

$$\begin{bmatrix}\frac{T}{\sigma}\frac{\varphi(\bar{\upsilon})-\varphi(\bar{\delta})}{(\Phi(\bar{\upsilon})-\Phi(\bar{\delta}))}+\frac{1}{\sigma^2}\sum_{t\in\tau}(r_t-\mu)\\ -\frac{T}{\sigma^2}\frac{\bar{\upsilon}\varphi(\bar{\upsilon})-\bar{\delta}\varphi(\bar{\delta})}{(\Phi(\bar{\upsilon})-\Phi(\bar{\delta}))}-\frac{T}{\sigma}+\frac{1}{\sigma^3}\sum_{t\in\tau}(r_t-\mu)^2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \quad (10)$$

However the nonlinearity of the system of equations obtained due to the presence of $\frac{\varphi(\bar{\upsilon})-\varphi(\bar{\delta})}{\Phi(\bar{\upsilon})-\Phi(\bar{\delta})}$ makes the solution quite complex and needs a numeric computational assistance. ¡

Cohen and Whitten (1988) gave an approach to find a solution and provided tables for this task. However we will use a non-linear optimization algorithm directly in the empirical part.

4.2 The Case of Symmetric Truncation

If $\bar{r}:=\frac{\sum r_t}{T}=\frac{\delta+\upsilon}{2}$, we consider $\hat{\mu}=\bar{r}$, and we get

$$\frac{\varphi(\bar{\upsilon})-\varphi(\bar{\delta})}{(\Phi(\bar{\upsilon})\quad\Phi(\bar{\delta}))}=0$$

As $\bar{\delta}$ and $\bar{\upsilon}$ becomes symmetric around 0 and thus$(\bar{\delta})=\varphi(\bar{\upsilon})$.

Equation 10 becomes: ¡

$$\begin{bmatrix} \frac{1}{\sigma^2}\sum_{t\in\tau}(r_t-\bar{r}) \\ -\frac{T}{\sigma^2}\frac{2\,\bar{\delta}\,\varphi(\bar{\delta})}{1-2\,\Phi(\bar{\delta})}-\frac{T}{\sigma}+\frac{1}{\sigma^3}\sum_{t\in\tau}(r_t-\bar{r})^2 \end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$-\frac{T}{\sigma}\frac{2\,\bar{\delta}\,\varphi(\bar{\delta})}{1-2\,\Phi(\bar{\delta})}-T+\frac{1}{\sigma^2}\sum_{t\in\tau}(r_t-\bar{r})^2=0$$

$$\Leftrightarrow\sum_{t\in\tau}(r_t-\bar{r})^2=\sigma^2\left(\frac{T}{\sigma}\frac{2\,\bar{\delta}\,\varphi(\bar{\delta})}{1-2\,\Phi(\bar{\delta})}+T\right)=T\sigma\frac{2\,\bar{\upsilon}\,\varphi(\bar{\upsilon})}{1-2\,\Phi(\bar{\upsilon})}+T\sigma^2,$$

Note 1. The statistical literature produced other estimators for the truncated normal distribution parameters based on the method of moments. However many studies confirmed its higher bias and lower efficiency compared to the MLE method.

5. Testing for Truncated Normality

To an extent, the observer can only view a specific range of the population which leads to truncated samples. In manufacturing for example, at the end of manufacturing lines, products that passed some quality check will only be observed. This fact will imply some deviation of stochastic characteristics of studied sample from its population (mean, standard deviation, distribution, etc...). Thus, using the usual statistical technics may lead to a misleading conclusion such as: misspecification of moments or rejecting/accepting normality.

The econometric literature has produced a normality test within a range, however the Jarque-Bera continued to be the most used test. The idea of this test is related to the fact that all normal distributions have the same skewness and excess kurtosis of zero, which is a distinguishing fact from other distributions of the Pearson family.

5.1 The Classic Jarque-Bera Test

Appendix 3 discussed the effects of truncation on the first and second moments of normal variable which becomes lower as the truncation range grows wider and in case of symmetric truncation around the mean. We can intuitively extend this result to higher order moments and in particular skewness and kurtosis. A truncated normal sample, on a large truncation range (compared to its standard deviation) can pass the JB test.

We have assessed the efficiency of the JB test through a series of simulations. In symmetric case

Table 1. Acceptance rate of H_0 of JB test for a sample following $\mathcal{N}_{-1}^{1}(0,1)$ at 5%

Size

l	10	100	1000
1	0.998	0.8293	0.0000
2	0.996	0.9991	0.0001

3	0.993	0.9928	0.9547
4	0.989	0.9610	0.9640
5	0.990	0.9563	0.9519
6	0.990	0.9550	0.9512
7	0.988	0.9555	0.9497
8	0.990	0.9604	0.9498
9	0.990	0.9629	0.9494
10	0.989	0.9599	0.9501

Table 2. Acceptance rate of H_0 of JB test for a sample following $\mathcal{N}_{-10}^{10}(0,1)$ at 5%

Size

l	10	100	1000
1	0.985	0.530	0.000
2	0.994	0.952	0.038
3	0.994	0.974	0.954
4	0.993	0.960	0.957
5	0.99	0.96	0.952
6	0.991	0.963	0.959
7	0.992	0.974	0.956
8	0.992	0.949	0.038
9	0.986	0.538	0.000
10	0.956	0.04	0.000

In conclusion, a symmetrically truncated normal sample has a great chance of passing the JB test of normality, only if the sample size remains small or medium, or, if the truncation range exceeds 6 times the standard deviation. Unlike range, asymmetry does not have a monotone impact on the chance of passing the JB test.

The next subsection will see the attempt to modify the ordinary JB test to enhance its efficiency in detecting normality in truncated samples.

5.2 The Truncated Pearson Distribution

This subsection will focus on the truncation of Pearson distributions.

Let $\mathcal{X}$ a random variable following a Pearson distribution $\mathcal{P}(\boldsymbol{\beta})$ (where $\boldsymbol{\beta} \in \mathbb{R}^4$), and let $f(x;\boldsymbol{\beta})$ denote its distribution density function. We have

$$\frac{f'(x;\boldsymbol{\beta})}{f(x;\boldsymbol{\beta})} = -\frac{x-\beta_0}{\beta_1+\beta_2 x+\beta_3 x^2} \tag{11}$$

Let X the truncation of $\mathcal{X}$ on $(a,b) \subset \mathbb{R}$ and $p(x;\boldsymbol{\beta})$ its distribution function. We have:

$$p(x;\boldsymbol{\beta}) = \begin{cases} c.f(x;\boldsymbol{\beta}) & x \in (a,b) \\ 0 & x \notin (a,b) \end{cases} \tag{12}$$

We can see that for $x \in (a,b)$, $p(x;\boldsymbol{\beta})$ verifies the differential equation 11 and we can write

$$\frac{p'(x;\boldsymbol{\beta})}{p(x;\boldsymbol{\beta})} = -\frac{x-\beta_0}{\beta_1+\beta_2 x+\beta_3 x^2} \quad \forall x \in (a,b)$$

Solving this equation gives

$$p(x;\boldsymbol{\beta}) = C(\boldsymbol{\beta})\exp\left(-\int \frac{x-\beta_0}{\beta_1+\beta_2 x+\beta_3 x^2}\,dx\right) \quad \forall x \in (a,b) \tag{13}$$

As $p(x;\boldsymbol{\beta})$ is a PDF, we find that

$$C(\boldsymbol{\beta}) = \left[\int_a^b \exp\left(-\int \frac{x-\beta_0}{\beta_1+\beta_2 x+\beta_3 x^2}\,dx\right)dx\right]^{-1}$$

Let

$$F(x;\boldsymbol{\beta}) := -\int_{\beta_0}^{x} \frac{z-\beta_0}{\beta_1+\beta_2 z+\beta_3 z^2}\,dz$$
$$g(x;\boldsymbol{\beta}) := \exp(F(x;\boldsymbol{\beta}))$$
$$G(\boldsymbol{\beta}) := \frac{1}{C(\boldsymbol{\beta})} = \int_a^b g(z,\boldsymbol{\beta})dz$$

We can rewrite 13 as follows:

$$p(x;\boldsymbol{\beta}) = \frac{g(x;\boldsymbol{\beta})}{G(\boldsymbol{\beta})} \quad \forall x \in (a,b)$$

5.3 The LM Test Methodology

5.3.1 The LM Test Statistic

Let $L(\boldsymbol{\beta})$ be the log-likelihood function of a $k \times 1$ vector of parameters and let $q(\boldsymbol{\beta})$ and $I(\boldsymbol{\beta})$ respectively the score and information matrix defined as follows:

$$\boldsymbol{q}(\beta) = \frac{\partial L(\beta)}{\partial \beta}$$

$$I(\boldsymbol{\beta}) = -\mathbb{E}\left(\frac{\partial^2 L(\boldsymbol{\beta})}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}'}\right)$$

Let $\tilde{\boldsymbol{\beta}}$ be the maximum likelihood estimator of βsubject to$r \times 1$ vectors of constraints $\boldsymbol{h}(\boldsymbol{\beta}) = 0$. If we consider the Lagrangian function

$$\mathcal{L}(\boldsymbol{\beta}, \boldsymbol{\lambda}) = L(\boldsymbol{\beta}) - \boldsymbol{\lambda}' \boldsymbol{h}(\boldsymbol{\beta})$$

Where $\boldsymbol{\lambda}$ *is* $r \times 1$ vector of Lagrange multipliers, the first order conditions for $\tilde{\boldsymbol{\beta}}$ are:

$$\frac{\partial \mathcal{L}(\boldsymbol{\beta}, \lambda)}{\partial \boldsymbol{\beta}} = \boldsymbol{q}(\tilde{\beta}) - \boldsymbol{\eta}(\tilde{\beta})\tilde{\boldsymbol{\lambda}} = 0$$
$$\frac{\partial \mathcal{L}(\boldsymbol{\beta}, \lambda)}{\partial \boldsymbol{\lambda}} = \boldsymbol{h}(\tilde{\boldsymbol{\beta}}) = 0$$

Where

$$\boldsymbol{\eta}(\beta) = \frac{\partial \boldsymbol{h}(\beta)'}{\partial \beta}$$

The test statistic is given by

$$LM = \tilde{q}' \tilde{I}^{-1} \tilde{q} = \tilde{\lambda}' \tilde{\eta} \tilde{I}^{-1} \tilde{\eta} \tilde{\lambda}$$

The LM statistic is asymptotically efficient and follows a χ^2 h, while using only the estimates of the parameter under the null hypothesis (unlike the LR test)

5.4 Testing Normality From Truncated Sample

5.4.1 The Normal Distribution as a Special Case of Pearson Distribution

Let $\varphi(x; \mu, \sigma)$ the $\mathcal{N}(\mu, \sigma)$ PDF, we have

$$\frac{\varphi'(x; \mu, \sigma)}{\varphi(x; \mu, \sigma)} = -\frac{x-\mu}{\sigma^2} = -\frac{x-\mu}{\sigma^2 + 0x + 0x^2}$$

Thus, the normal distribution is a special case of Pearson distribution with $\beta = [\mu, \sigma^2, 0, 0]$.

This idea was based on the Jacque-Bera test of normality using the LM methodology. However, the JB test statistics is unable to be taken into consideration due to the effect of truncation on population moments.

As established above, truncation does not change the parameters of the distribution, but it changes the PDF thus the likelihood function. Because the LM test is based on the log-likelihood function, it's expected that the test statistic will change in the case of truncation. The Truncated Jarque-Bera (TJB) test that is being developed here is based on the LM methodology.

5.4.2 The TJB Test Statistic

Let $\{X_i\}_{i\in I}\ i = 1 \dots n$ a sequence of iid random variable distributed according to a truncated

$$X_i \sim \mathcal{P}_a^b(\beta_0, \beta_1, \beta_2, \beta_3) \quad i \in I$$

Pearson distribution:

Let $\theta_0 = (\beta_0, \beta_1)$ and $\theta_2 = (\beta_2, \beta_3)$

$$X_i \sim \mathcal{N}_a^b(\mu, \sigma^2) \Leftrightarrow \begin{cases} \boldsymbol{\theta}_0 = (\mu, \sigma^2) \\ \boldsymbol{\theta}_1 = 0 \end{cases}$$

$$\begin{cases} H_0\colon \boldsymbol{\theta}_1 = 0 \\ H_a\colon \boldsymbol{\theta}_1 \neq 0 \end{cases}$$

$$p_i(x_i) = p(x_i) = \frac{g(x_i; \boldsymbol{\beta})}{G(\boldsymbol{\beta})} \quad x \in (a, b)$$

Thus the hypothesis test

We have

The log-likelihood function is thus:

$$\ell(\{x_i\}_{i\in I}; \beta) \ = \sum_{i\in I} [\log(p(x_i; \beta)) - \log(G(\beta))]$$

We should note here that:

$$\boldsymbol{h}(\beta) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \beta \Rightarrow \boldsymbol{\eta}(\beta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For $\boldsymbol{\beta} = \widetilde{\boldsymbol{\beta}}$, we have:

$$\boldsymbol{q}(\{x_i\}_{i\in I}; \tilde{\beta}) - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{\lambda} = 0 \Rightarrow \begin{bmatrix} \ell_0'(\{x_i\}_{i\in I}; \tilde{\beta}) \\ \ell_1'(\{x_i\}_{i\in I}; \tilde{\beta}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{14}$$

The result found in 14, implies the following:

$$\begin{cases} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \hat{\mu} \\ \hat{\sigma}^2 \end{bmatrix} \\ \text{TJB} = \tilde{\boldsymbol{\chi}}'[\tilde{\nabla}_{22} - \tilde{\nabla}_{21}\tilde{\nabla}_{11}^{-1}\tilde{\nabla}_{12}]_{\sim} \end{cases}$$

With $\hat{\mu}$ and $\hat{\sigma}$ are the MLE estimators respectively of μ and σ

$$\tilde{\chi} = \begin{bmatrix} \ell_2'(\{x_i\}_{i\in I}; \tilde{\beta}) \\ \ell_3'(\{x_i\}_{i\in I}; \tilde{\beta}) \end{bmatrix}$$

$$\tilde{\nabla}_{ij} = \frac{\partial l(\beta)}{\partial \theta_i' \partial \theta_j}\bigg|_{\tilde{\beta}}$$

The derivation of the TJB statistic is done as follows:

1. Calculate the ML estimator of μ and σ
2. Calculate $\tilde{X}$
3. Calculate $\tilde{\nabla}_{22}, \tilde{\nabla}_{21}, \tilde{\nabla}_{11}^{-1}, \tilde{\nabla}_{12}$
4. Calculate the TJB statistic by matrix operations using results form 2 and 3
5. Calculate the associated p-value using the $\chi^2(2)$ distribution

The detailed calculations and algorithm are given in appendix A.

5.4.3 Convergence Issues

In order to assess the effect of size, truncation range and centricity on the TJB statistic convergence, we have conducted several simulations (with 100,000 repetition each). The results are resumed below

Impact of sample size

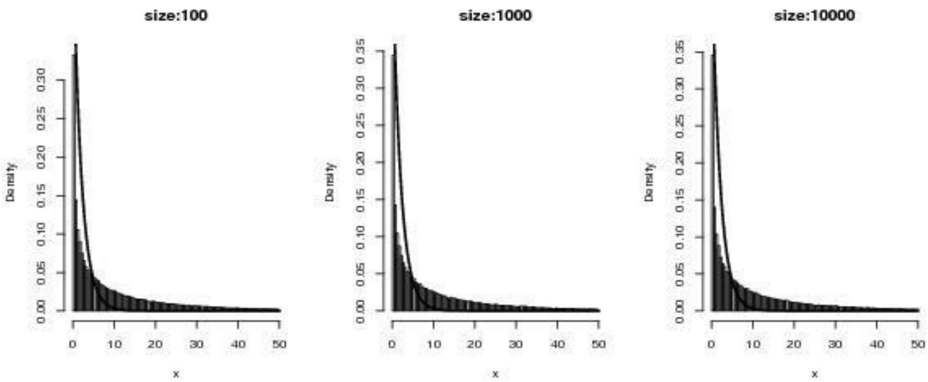


Figure 5. TJB statistic distribution for different sample sizes (simulated from $\mathcal{N}_{-1}^{1}(0,1)$) compared to $\chi^2(2)$ distribution

We can remark here the slow convergence of the TJB statistic to the $\chi^2(2)$ distribution. However, more than 70% of trials passes the truncated normality test using the p-values of the asymptotic distribution.

The impact of truncation range

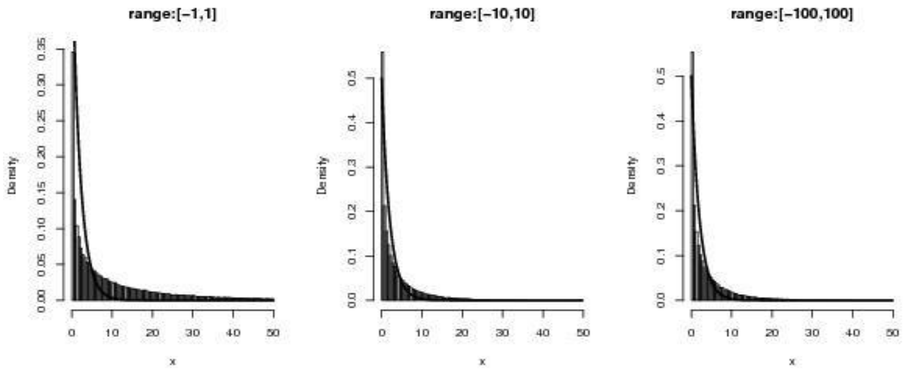


Figure 6. TJB statistic distribution for different truncation ranges (simulated from $\mathcal{N}(0,1)$ truncated to the range) compared to $\chi^2(2)$ distribution

We can remark here that the speed of convergence of the TJB statistic to the $\chi^2(2)$ distribution increases with the range length (while maintaining the centricity i.e. the mean being the center of the range). For the range [-10,10] (and similarly for [-100,100]), about 90% of trials passes the truncated normality test using the p-values of the asymptotic distribution.

The impact of centricity

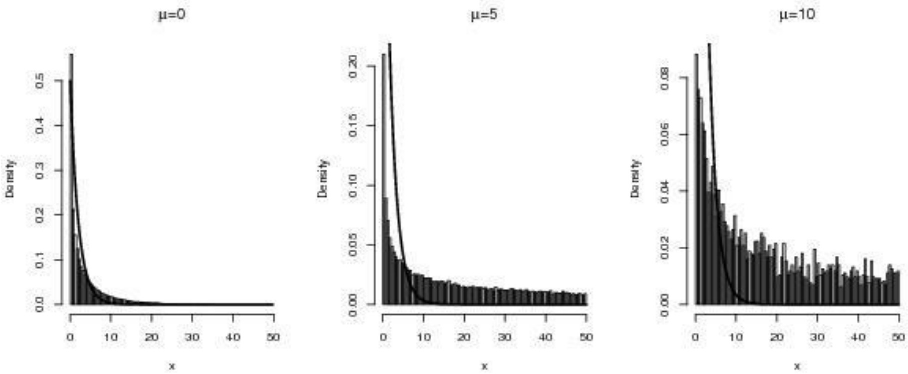


Figure 7. TJB statistic distribution for different truncation ranges (simulated from a 1000 size sample of $\mathcal{N}_{-1}^{1}(0,1)$) compared to $\chi^2(2)$ distribution

We can remark here that the speed of convergence is extremely sensitive to the position of μ in the truncated range, in fact, for the same sample size and range. For the range [-10,10] (and similarly for [-10,10]), about 90% of trials passes the truncated normality test using the p-values of the asymptotic distribution when μ is the center of the range versus only 13% when μ is at one extremity of the range.

Remark 1. The slow convergence of the TJB statistic to $\chi^2(2)$ distribution can be overcame by creating specific distribution tables. However, a substantial statistical power loss may arise as the truncated normal distribution in tight ranges becomes very similar to the other truncated fat tailed symmetric distribution such as the Cauchy distribution:

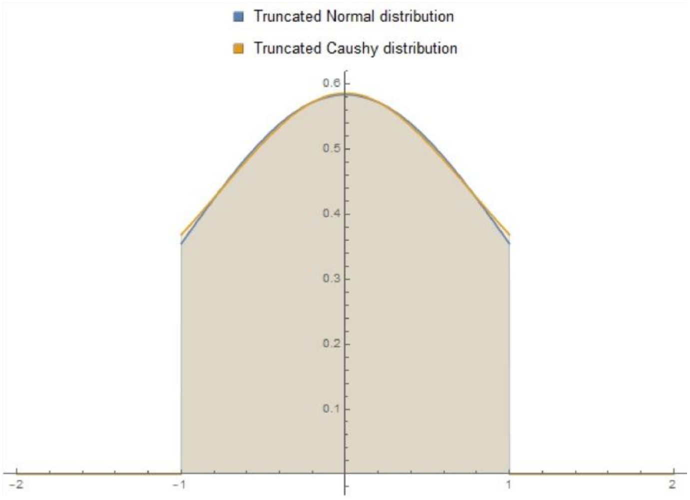


Figure 8. [-1, 1] truncated Normal and Cauchy distributions

6. Application to the Tunisian Stock Exchange

Owned by the 23 brokers, the TSE is positioned in the heart of the Tunisian financial system which contains the brokers that represent the trading monopoles, the financial market council as the legal authority supervising the financial system of the country and the guarantee funds which job is to protect investors from various risks. The Tunis Stock Exchange is made of the principal market which contains the listed big companies, the alternative market in which small and medium size firms are listed, the bond marketand the hors cote market (designed for unlisted firms that desiring financing).

The normal trading hours start at 9 am and end at 2 pm, with a pre-opening session from 9 to 10 am and a pre-closing session from 2 to 2:05 pm. Trading has to be within a 3% window of the previous closing price. Once the price of a stock hits this limit, its trading is stopped for 15 minutes. As a result, the ceiling and floor are increased by an extra 1.5% until the limit of 6.09%of yesterday's closing price is reached.

6.1 A First Look to the Data

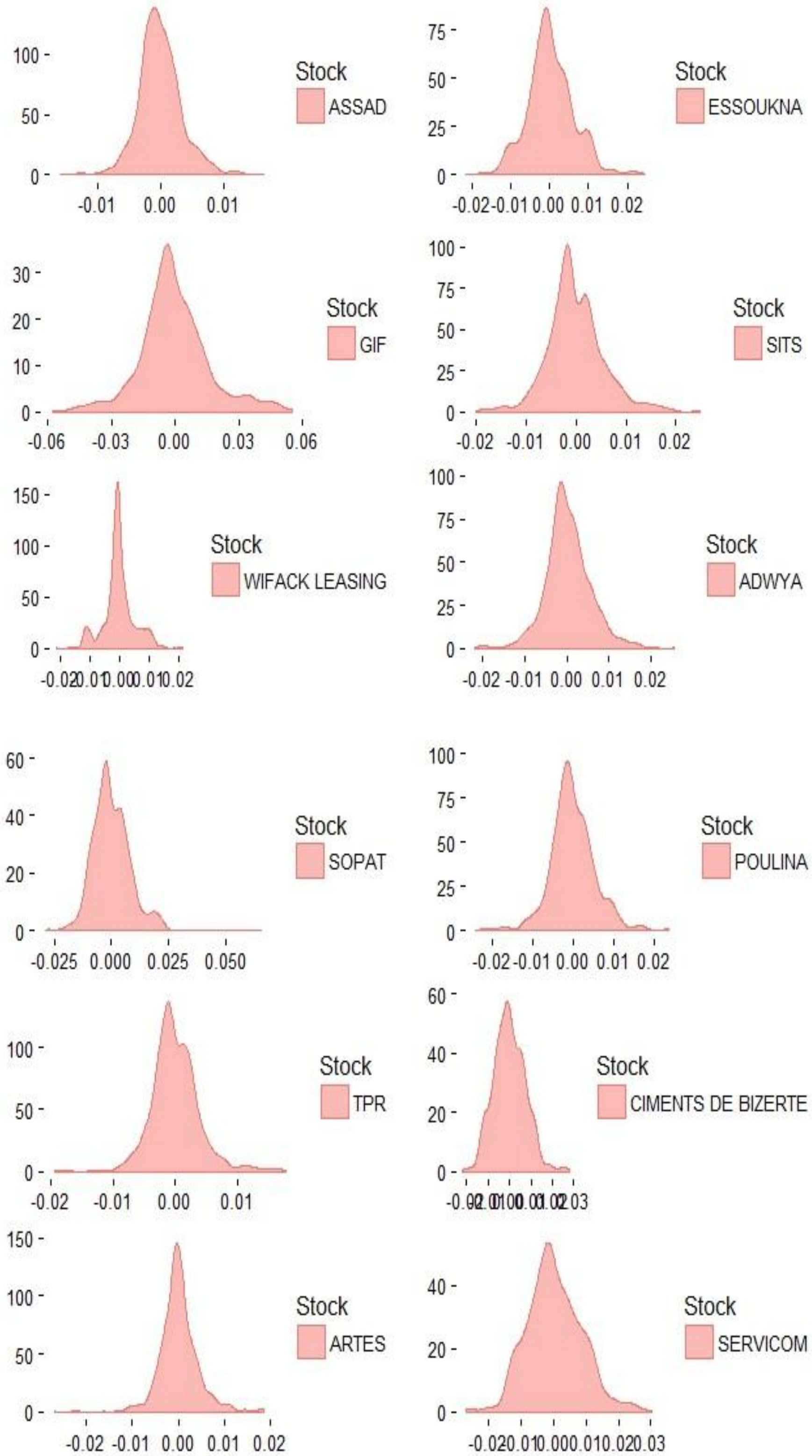
The 17 stocks included in the empirical study were chosen based on two criteria:

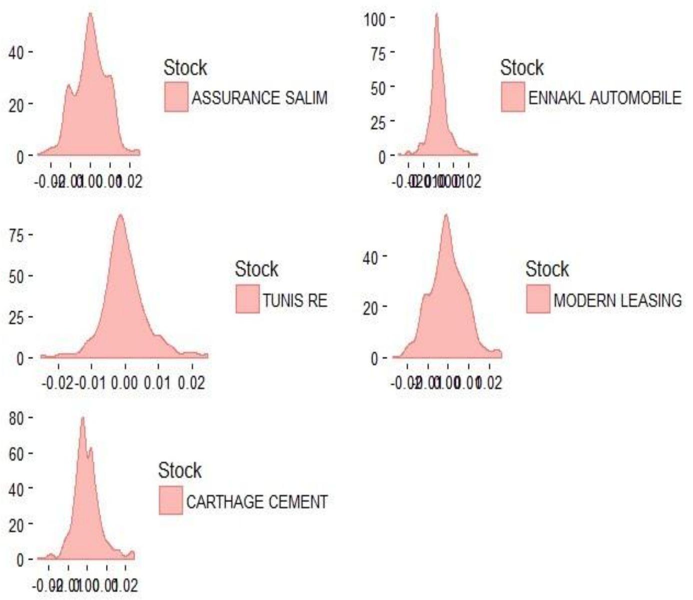
- Liquidity: The stocks with high liquidity have more tendency to hit the limit barriers due to higher volume of transactions
- Date of introduction: As the empirical study was done in 2017, only the stocks that were introduced before 2013 were considered to provide 1000 data points.

The sample covers the period 2013 - 2017.

International Journal of Accounting and Financial Reporting
ISSN 2162-3082
2019, Vol. 9, No. 3

6.1.1 Log-Returns Distribution





6.1.2 Log-Returns Main Statistics

We display below the main statistics of the selected stocks along with the JB statistic.

Table 3. Main statistics of daily log-returns

	Mean	SD	Skewness	Kurtosis
ASSAD	0	0.003	0.290	5.331
GIF	0	0.016	0.381	4.361
WIFACK LEASING	0	0.005	0.141	4.473
ESSOUKNA	0	0.006	0.334	4.102
SITS	0	0.006	0.393	4.592
ADWYA	0	0.006	0.185	4.989
SOPAT	0	0.008	0.922	8.122
TPR	0	0.004	0.413	6.730
ARTES	0	0.004	-0.075	8.377
POULINA	0	0.006	0.142	5.339

CIMENTS DE BIZERTE	0	0.007	0.317	3.420
SERVICOM	0	0.009	0.283	3.496
ASSURANCE SALIM	0	0.008	-0.078	3.036
TUNIS RE	0	0.007	0.467	5.274
CARTHAGE CEMENT	0	0.007	0.561	5.155
ENNAKL AUTOMOBILE	0	0.006	0.196	5.955
MODERN LEASING	0	0.008	0.187	3.216

6.2 Use of Truncation Method

Table 4. MLE estimation results for daily log-returns

	^	^
ASSAD	0	0.000
GIF	0	0.000
WIFACK LEASING	0	0.001
ESSOUKNA	0	0.001
SITS	0	0.000
ADWYA	0	0.000
SOPAT	0	0.000
TPR	0	0.000
ARTES	0	0.000
POULINA	0	0.000

CIMENTS DE BIZERTE	0	0.000
SERVICOM	0	0.001
ASSURANCE SALIM	0	0.001
TUNIS RE	0	0.001
CARTHAGE CEMENT	0	0.000
ENNAKL AUTOMOBILE	0	0.001
MODERN LEASING	0	0.001

The extreme values obtained here are mainly caused by the numerical computation of ML estimators of β_0 and β_1 (a detailed assessment is given in appendix B). This is solved by using the symmetric truncation ML estimator as in TJB0 statistic (The use of sample mean and variance will have a small bias as it was remarked that the log(Return) mean is close to the center of truncation range ([log(1-0.069),log(1+0.069)]) which is large compared to the sample standard deviation.

Table 5. Three tests results for daily log-returns normality

Stocks	JB	TJB	TJB0
ASSAD	198.029	0.000000e+00	184.526
GIF	90.109	2.478479e+13	74.669
WIFACK LEASING	69.154	5.469668e+08	65.618
ESSOUKNA	54.181	4.317395e+09	38.794
SITS	106.459	9.070122e+10	84.347
ADWYA	141.890	2.389226e+14	135.475
SOPAT	1017.540	0.000000e+00	893.870
TPR	490.258	3.305068e+10	463.010
ARTES	972.870	2.321727e+12	964.586

POULINA	186.480	3.581983e+09	181.708
CIMENTS DE BIZERTE	17.487	2.901209e+09	5.089
SERVICOM	19.164	7.813152e+10	8.039
ASSURANCE SALIM	0.667	0.000000e+00	0.018
TUNIS RE	216.365	7.964002e+10	183.074
CARTHAGE CEMENT	203.430	2.774102e+12	158.154
ENNAKL AUTOMOBILE	292.572	7.272689e+13	284.540
MODERN LEASING	5.971	8.005355e+10	1.365

The $\widetilde{JB}$ was significantly smaller for all the 17 daily log-returns. This mean that the modified JB test statistic succeeded in capturing the impact of truncation on the skewness and kurtosis. However no other daily log-return succeeded the $\widetilde{JB}$test than the ones that already passed the JB test.

7. Conclusion

This paper addressed the issue of microstructure effects due to price limits. In particular, stocks returns showed a non-normal behavior in the case of price limits. This non-normality could potentially be coming from a change in investors behavior, valuation effect, a change in the stock returns statistical properties, value effect. Both effects occur as a consequence to price limitations: Unconditional equilibrium prices shadowed prices is unobservable by agents due to the fact that the asset valuation will be guided by the limited future prices assumption and the conditional equilibrium prices shadowing prices which are not observed by agents because they can only transact within a limited range. Thus, the estimation of the fair value of the asset will be complex and the existing trading strategies that focus only on observed prices will be inefficient. In order to analyze this joint-hypothesis, a new normality test is defined and takes into account the truncation effect, the TJB test.

Our results demonstrated that most daily stock returns studied in this paper failed to pass the JB and TJB test. Hence the first hypothesis was confirmed as the stock returns are not normal. Additionally, a discovery was made: in the case of log-normal daily returns (and potentially other thin tailed distributions) taking into account the effect of truncation is of limited interest if the truncation range is wide enough and symmetric around the mean.

Lastly, our results demonstrated a weak Value effect. This allowed us to think that the main behavior discrepancy comes from Valuation effect. We do think that the price limits set a psychological barrier for stock exchange agents for what they believe is a 'fair price'. Further studies are needed to analyze this behavior.

References

Bachelier, L. (1900). *Théorie de la spéculation*. Gauthier-Villars.

Bliss, C. I., & Stevens, W. L. (1937). The calculation of the time-mortality curve. *Annals of Applied Biology*, *24*(4), 815-852.

Chen, G.-M., Rui, O. M., & Wang, S. S. Y. (2005). The effectiveness of price limits and stock characteristics: evidence from the Shanghai and Shenzhen stock exchanges. *Review of Quantitative Finance and Accounting*, *25*(2), 159-182.

Cohen, A. C. (1959). Simplified estimators for the normal distribution when samples are singly censored or truncated. *Technometrics*, *1*(3), 217-237.

Cohen, A. C. Jr. (1950). Estimating the mean and variance of normal populations from singly truncated and doubly truncated samples. *The Annals of Mathematical Statistics*, 557-569.

Cohen, A. C., & Whitten, B. J. (1988). *Parameter estimation in reliability and life span models*. M. Dekker.

Cowles, A. (1960). A revision of previous conclusions regarding stock price behavior. *Econometrica: Journal of the Econometric Society*, 909-915.

Errais, E., & Bahri, D. (2016). Is standard deviation a good measure of volatility? The case of African markets with price limits. *Annals of Economics & Finance*, *17*(1).

Fama, E. F. (1963). Mandelbrot and the stable Paretian hypothesis. *The Journal of Business*, *36*(4), 420-429.

Fama, E. F. (1965). The behavior of stock-market prices. *The Journal of Business*, *38*(1), 34-105.

Fama, E. F. (1989). Perspective on October 1987, or what did we learn from the crash?. *Black Monday and the Future of the Financial Markets*, Irwin, Homewood, III.

Fisher, R. A. (1931). The truncated normal distribution. *British Association for the Advancement of Science, Mathematical Tables*, *1*.

Galton, F. (1997). An examination into the registered speeds of American trotting horses, with remarks on their value as hereditary data. *Proceedings of the Royal Society of London*, *62*(379-387), 310-315.

Greene, W. H. (2003). *Econometric analysis*. Pearson Education India.

Grossman, S. F., & Stiglitz, J. E. (1980). On the impossibility of informationally efficient markets. *The American Economic Review*, *70*(3), 393-408.

Jarque, C. M., & Bera, A. K. (1980). Efficient tests for normality, homoscedasticity and serial independence of regression residuals. *Economics Letters*, *6*(3), 255-259.

Kahenman, D., & Tversky, A. (1979). Prospect theory: an analysis of decision under risk. *Econometrica: Journal of the Econometric Society*, 263-291.

Kendall, M. G., & Hill, A. B. (1953). The analysis of economic time-series part i: prices. *Journal of the Royal Statistical Society: Series A (General)*, *116*(1), 11-34.

Kim, K. A., & Rhee, S. (1997). Price limit performance: evidence from the Tokyo stock exchange. *The Journal of Finance*, *52*(2), 885-901.

Larson, A. B. (1960). Measurement of a random process in futures prices. *Proceedings of the Annual Meeting (Western Farm Economics Association)*, *33*, 101-112.

Lauterbach, B., & Ben-Zion, U. (1993). Stock market crashes and the performance of circuit breakers: empirical evidence. *The Journal of Finance*, *48*(5), 1909-1925.

Leroy, S. F., & Porter, R. D. (1981). The present-value relation: tests based on implied variance bounds. *Econometrica: Journal of the Econometric Society*, 555-574.

Lux, T. (1998). The socio-economic dynamics of speculative markets: interacting agents, chaos, and the fat tails of return distributions. *Journal of Economic Behavior & Organization*, *33*(2), 143-165.

Malkiel, B. G., & Fama, E. F, (1970). Efficient capital markets: a review of theory and empirical work. *The Journal of Finance*, *25*(2), 383-417.

Mandelbrot, B. (1962). Paretian distributions and income maximization. *The Quarterly Journal of Economics*, *76*(1), 57-85.

Mehra, R. (2012). Consumption-based asset pricing models. *Annu. Rev. Financ. Econ.*, *4*(1), 385-409.

Orjebin, E. (2014). *A recursive formula for the moments of a truncated univariate normal distribution*.

Osborne, M. F. M. (1959). Brownian motion in the stock market. *Operations Research*, *7*(2), 145-173.

Pearson, K. (1902). Note on Francis Galton's problem. *Biometrika*, *1*(4), 390-99.

Pearson, K., & Lee, A. (1908). On the generalized probable error in multiple normal correlation. *Biometrika*, *6*(1), 59-68.

Peters, E. E. (1994). *Fractal market analysis: applying chaos theory to investment and economics* (volume 24). John Wiley & Sons.

Roberts, H. V. (1967). *Statistical versus clinical prediction of the stock market*.

Samuelson, P. A. (1965). Proof that properly anticipated prices fluctuate randomly. *IMR; Industrial Management Review (pre-1986)*, *6*(2), 41.

Shiller, R. J. (1981). The use of volatility measures in assessing market efficiency. *The Journal of Finance*, *36*(2), 291-304.

Tsay, R. S. (2005). *Analysis of financial time series* (volume 543). John Wiley & Sons.

Wang, D. Y., Chong, T. T.-L., & Chan, W. H. (2014). *Price limits and stock market volatility in china*.

Appendix

1. Calculation of the LM Statistic

We have for $j,k = 0 \ldots 3$:

$$\ell'_j(\{x_i\}_{i\in I};\tilde{\boldsymbol{\beta}}) = \sum_{i\in I} F'_j(x_i,\tilde{\boldsymbol{\beta}}) - N\,\frac{G'_j(\tilde{\boldsymbol{\beta}})}{G(\tilde{\boldsymbol{\beta}})}$$
$$\ell''_{j,k}(\{x_i\}_{i\in I};\tilde{\boldsymbol{\beta}}) = \sum_{i\in I} F''_{j,k}(x_i,\tilde{\boldsymbol{\beta}}) - N\,\frac{(G''_{j,k}.G)(\tilde{\boldsymbol{\beta}}) - (G'_j.G'_k)(\tilde{\boldsymbol{\beta}})}{G^2(\tilde{\boldsymbol{\beta}})}$$

In order to calculate the LM statistic we need to derive the gradient and hessian of F and G and some moments of $X \sim \mathcal{N}_a^b(\tilde{\beta}_0, \tilde{\beta}_1)$

1.1 Calculation of Gradient and Hessian of $F(x;\tilde{\beta})$

We have:

$$F(x,\boldsymbol{\beta}) = -\int_{\beta_0}^{x} \frac{z-\beta_0}{\beta_1+\beta_2 z+\beta_3 z^2}\,dz$$

1.1.1 Case $\beta_2 = \beta_3 = 0$

$$\int_{\beta_0}^{x} \frac{z-\beta_0}{\beta_1}\,dz = \frac{1}{\beta_1}\left[\frac{z^2}{2} - \beta_o z\right]_{\beta_0}^{x}$$

$$F'_0(x;\tilde{\boldsymbol{\beta}}) = -\frac{\partial}{\partial\beta_0}\frac{1}{\beta_1}\left[\frac{z^2}{2}-\beta_o z\right]_{\beta_0}^{x}\Bigg|_{(\beta_0,\beta_1)=(\tilde{\beta}_0,\tilde{\beta}_1)} = \frac{x-\tilde{\beta}_0}{\tilde{\beta}_1}$$
$$F'_1(x;\tilde{\boldsymbol{\beta}}) = -\frac{\partial}{\partial\beta_0}\frac{1}{\beta_1}\left[\frac{z^2}{2}-\beta_o z\right]_{\beta_0}^{x}\Bigg|_{(\beta_0,\beta_1)=(\tilde{\beta}_0,\tilde{\beta}_1)} = \frac{(x-\tilde{\beta}_0)^2}{2\tilde{\beta}_1^2}$$
$$F''_{00}(x;\tilde{\boldsymbol{\beta}}) = \frac{\partial}{\partial\beta_0}\frac{x-\beta_0}{\beta_1}\Bigg|_{(\beta_0,\beta_1)=(\tilde{\beta}_0,\tilde{\beta}_1)} = -\frac{1}{\tilde{\beta}_1}$$
$$F''_{01}(x;\tilde{\boldsymbol{\beta}}) = \frac{\partial}{\partial\beta_1}\frac{x-\beta_0}{\beta_1}\Bigg|_{(\beta_0,\beta_1)=(\tilde{\beta}_0,\tilde{\beta}_1)} = -\frac{x-\tilde{\beta}_0}{\tilde{\beta}_1^2}$$
$$F''_{11}(x;\tilde{\boldsymbol{\beta}}) = \frac{\partial}{\partial\beta_1}\frac{(x-\beta_0)^2}{2\beta_1^2}\Bigg|_{(\beta_0,\beta_1)=(\tilde{\beta}_0,\tilde{\beta}_1)} = -\frac{(x-\tilde{\beta}_0)^2}{\tilde{\beta}_1^3}$$

1.1.2 Case $\beta_3 = 0$

$$\int_{\beta_0}^{x} \frac{z-\beta_0}{\beta_1+\beta_2 z}\,dz = \int_{\beta_0}^{x} a + \frac{b}{\beta_1+\beta_2 z}\,dz = [a z - b\log(|\beta_1+\beta_2 z|)]_{\beta_0}^{x}$$

with

$$a = \frac{1}{\beta_2}$$
$$b = \beta_0 - \frac{\beta_1}{\beta_2}$$

$$\begin{aligned}
F_2'(x;\tilde{\boldsymbol{\beta}}) &= -\lim_{\beta_2\to 0}\frac{\partial}{\partial\beta_2}[az - b\log(|\beta_1+\beta_2 z|)]_{\beta_0}^{x}\Big|_{(\beta_0,\beta_1)=(\tilde\beta_0,\tilde\beta_1)} = \frac{2x^3-3\tilde\beta_0x^2+\tilde\beta_0^3}{6\,\tilde\beta_1^2}\\
F_{20}''(x;\tilde{\boldsymbol{\beta}}) &= \frac{\partial}{\partial\beta_0}\frac{2x^3-3\beta_0x^2+\beta_0^3}{6\,\beta_1^2}\Big|_{(\beta_0,\beta_1)=(\tilde\beta_0,\tilde\beta_1)} = -\frac{x^2-\tilde\beta_0^2}{2\tilde\beta_1^2}\\
F_{21}''(x;\tilde{\boldsymbol{\beta}}) &= \frac{\partial}{\partial\beta_1}\frac{2x^3-3\beta_0x^2+\beta_0^3}{6\,\beta_1^2}\Big|_{(\beta_0,\beta_1)=(\tilde\beta_0,\tilde\beta_1)} = -\frac{2x^3-3\tilde\beta_0x^2+\tilde\beta_0^3}{3\,\tilde\beta_1^2}\\
F_{22}''(x;\tilde{\boldsymbol{\beta}}) &= -\lim_{\beta_2\to 0}\frac{\partial^2}{\partial\beta_2^2}[az - b\log(|\beta_1+\beta_2 z|)]_{\beta_0}^{x}\Big|_{(\beta_0,\beta_1)=(\tilde\beta_0,\tilde\beta_1)} = -\frac{3x^4-4\beta_0x^3+\beta_0^4}{6\beta_1^3}
\end{aligned}$$

1.1.3 Case $\beta_3 \neq 0$

Let

$$\mathcal{A} := \{(\beta_1,\beta_2,\beta_3)\in\mathbb{R}^3 | \beta_2^2 \geqslant 4\beta_1\beta_2\}$$

We can prove that $\mathcal{A}$ is a neighborhood of $(\beta_0,\beta_1,\beta_2)$. For

$$(\beta_0,\beta_1,\beta_2)\in\mathcal{A}/\{(\beta_1,\beta_2,\beta_3)\in\mathbb{R}^3|\beta_3\neq 0\}$$

We have:

$$\begin{aligned}
\int_{\beta_0}^{x}\frac{z-\beta_0}{\beta_1+\beta_2z+\beta_3z^2}dz &= \int_{\beta_0}^{x}\frac{a}{\beta_3(z-z_1)}+\frac{b}{(z-z_2)}dz\\
&= \left[\frac{a}{\beta_3}\log(|z-z_1|)+b\log(|z-z_2|)\right]_{\beta_0}^{x}
\end{aligned}$$

$$\begin{aligned}
F_3'(x;\tilde{\boldsymbol{\beta}}) &= -\lim_{(\beta_2,\beta_3)\to 0}\frac{\partial}{\partial\beta_3}\left[\frac{a}{\beta_3}\log(|z-z_1|)+b\log(|z-z_2|)\right]\Big|_{(\beta_0,\beta_1)=(\tilde\beta_0,\tilde\beta_1)}\\
&= \frac{3x^4-4\tilde\beta_0x^3+\tilde\beta_0^4}{12\tilde\beta_1^2}\\
F_{30}''(x;\tilde{\boldsymbol{\beta}}) &= \frac{\partial}{\partial\beta_0}\frac{3x^4-4\beta_0x^3+\beta_0^4}{12\beta_1^2}\Big|_{(\beta_0,\beta_1)=(\tilde\beta_0,\tilde\beta_1)} = -\frac{x^3-\tilde\beta_0^3}{3\beta_1^2}\\
F_{31}''(x;\tilde{\boldsymbol{\beta}}) &= \frac{\partial}{\partial\beta_1}\frac{3x^4-4\beta_0x^3+\beta_0^4}{12\beta_1^2}\Big|_{(\beta_0,\beta_1)=(\tilde\beta_0,\tilde\beta_1)} = -\frac{3x^4-4\tilde\beta_0x^3+\tilde\beta^4}{6\tilde\beta_1^3}\\
F_{32}''(x;\tilde{\boldsymbol{\beta}}) &= -\lim_{(\beta_2,\beta_3)\to 0}\frac{\partial^2}{\partial\beta_3\partial\beta_2}\left[\frac{a}{\beta_3}\log(|z-z_1|)+\mathrm{b}\log(|z-z_2|)\right]\Big|_{(\beta_0,\beta_1)=(\tilde\beta_0,\tilde\beta_1)}\\
&= -\frac{4x^5-5\tilde\beta_0x^4+\tilde\beta_0^5}{10\,\tilde\beta_1^3}\\
F_{33}''(x;\tilde{\boldsymbol{\beta}}) &= -\lim_{(\beta_2,\beta_3)\to 0}\frac{\partial^2}{\partial\beta_3^2}\left[\frac{a}{\beta_3}\log(|z-z_1|)+\mathrm{b}\log(|z-z_2|)\right]\Big|_{(\beta_0,\beta_1)=(\tilde\beta_0,\tilde\beta_1)}\\
&= -\frac{5x^6-6\tilde\beta_0x^5+\tilde\beta_0^6}{15\tilde\beta_1^3}
\end{aligned}$$

1.2 Calculation of Gradient and Hessian of $G(\tilde\beta)$

We have:

$$G_i'(\boldsymbol{\beta}) = \frac{\partial}{\partial\beta_i}\int_a^b \exp(F(x,\boldsymbol{\beta}))dx$$

Let

$$\mathcal{A}_T := \{\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2, \beta_3) \in \mathbb{R}^4 | \beta_2^4 \geqslant 4\beta_1\beta_3\}$$
$$\mathcal{B}_T := \{(z; \boldsymbol{\beta}) \in \mathbb{R}^5 | \boldsymbol{\beta} \in \mathcal{A}_T \wedge z \in]z_1, z_2[\}$$

Where

$$z_1 = \min(\mathrm{Sol}(\beta_1 + \beta_2 z + \beta_3 z^3))$$
$$z_2 = \max(\mathrm{Sol}(\beta_1 + \beta_2 z + \beta_3 z^3))$$

Let us denote now

$$\mathcal{A}(a,b) = \{(\beta_0, \beta_1, \beta_2, \beta_3) \in \mathcal{A}_T | z_1 \leqslant a \wedge z_2 \geqslant b\}$$
$$\mathcal{B}(a,b) := \{(z; \boldsymbol{\beta}) \in | \boldsymbol{\beta} \in \mathcal{A}(a,b) \wedge z \in]z_1, z_2[\}$$

We can prove that $\mathcal{A}(a;b)$ is a neighborhood of $(\beta_0, \beta_1, 0{,}0)$.

For

$$\forall (z, \boldsymbol{\beta}) \in \mathcal{B}(a,b),\ \beta_1 + \beta_2 z + \beta_3 z^2 \neq 0$$

Thus,

$$z \rightarrow -\frac{z - \beta_0}{\beta_1 + \beta_2 z + \beta_3 z^2} \in C^\infty(]a,b[) \quad \forall (\beta_0, \beta_1, \beta_2, \beta_3) \in \mathcal{A}(a,b)$$

This implies that for a $(\beta_0, \beta_1, \beta_2, \beta_3) \in \mathcal{A}(a,b)$,

$$x \rightarrow \int_0^x \exp\left(-\int_{\beta_0}^t \frac{z - \beta_0}{\beta_1 + \beta_2 z + \beta_3 z^2}\right) dt \in C^\infty(]a,b[)$$

If *a* and *b* are both finite, we can write, for $\boldsymbol{\beta} \in \mathcal{A}(a;b)$,

$$G_i'(\boldsymbol{\beta}) = \frac{\partial}{\partial \beta_i} \int_a^b \exp(F(x,\boldsymbol{\beta}))dx = \int_a^b \frac{\partial}{\partial \beta_i} \exp(F(x,\boldsymbol{\beta}))dx$$

As $\tilde{\boldsymbol{\beta}} \in \mathcal{A}(a;b)$,, we have

$$G_i'(\tilde{\boldsymbol{\beta}}) = \int_a^b (F_i'.g)(x, \tilde{\boldsymbol{\beta}}))dx$$

With a similar construction, we can prove that

$$G_{ij}''(\tilde{\boldsymbol{\beta}}) = \int_a^b [(F_{ij}''.g) + (F_i'.F_j.g)](x, \boldsymbol{\beta})dx$$

As all $F_i'(\tilde{\boldsymbol{\beta}})$ and $F_{ij}''(\tilde{\boldsymbol{\beta}})$ are polynomial of *x*, it is possible to find an explicit formulae for $G_i'(\tilde{\boldsymbol{\beta}})$ and $G_{ij}''(\tilde{\boldsymbol{\beta}})$. However these formulae are very complicated, we will use an integration algorithm to calculate the LM statistic.

1.3 Moments of Truncated Normal Distribution

Like the case of normal distribution, the moments of truncated normal distribution can be given in a recursive formula. Let μ_k denote the *k*-th moment of an $X \sim \mathcal{N}_a^b(\beta_0, \beta_1)$:

$$\mu_k = \int_a^b x^k \frac{\varphi_{\beta_0,\beta_1}(x)}{\Phi_{\beta_0,\beta_1}(b) - \Phi_{\beta_0,\beta_1}(a)} dx$$

Where $\varphi_{\beta_0,\beta_1}$ and $\varphi_{\beta_0,\beta_1}$ denotes respectively the PDF and CDF of $X \sim \mathcal{N}(\beta_0, \beta_1)$. We have:

$$\mu_k = \sum_{i=0}^{k} \binom{k}{i} \sqrt{\beta_1}^{\,i} \beta_0^{k-i} L_i$$

where L_i is defined as follows:

$$L_i := \begin{cases} 1 & i=0 \\ -\dfrac{\varphi_{0,1}(\bar{b}) - \varphi_{0.1}(\bar{a})}{\Phi_{0,1}(\bar{b}) - \Phi_{0.1}(\bar{a})} & i=1 \\ -\dfrac{\bar{b}^{i-1}\varphi_{0,1}(\bar{b}) - \bar{a}^{i-1}\varphi_{0.1}(\bar{a})}{\Phi_{0,1}(\bar{b}) - \Phi_{0.1}(\bar{a})} + (i-1)L_{i-2} & i>1 \end{cases}$$

With

$$\bar{\bullet} := \frac{\bullet - \beta_0}{\sqrt{\beta_1}}.$$

1.4 Calculation Script (R Language)

Algorithm 1

```
LMT_trunorm <- function(data, b0, b1, down = min(data), up = max(data)) { library(tmvtnorm) N = length(data) if (is.na(b0) || is.na(b1)) { mle.fit <- mle.tmvnorm(data, down, up) if (is.na(b0)) { b0 = mle.fit@coef[[1]] }

if (is.na(b1)) { b1 = mle.fit@coef[[2]] ^ 2 } }

F1 <- function(i, x, b0, b1) { switch(i + 1,

(x - b0) / b1,

(x - b0) / (2 * b1 ^ 2),

(2 * x ^ 3 - 3 * b0 * x ^ 2 + b0 ^ 3) / (6 * b1 ^ 2),

(3 * x ^ 4 - 4 * b0 * x ^ 3 + b0 ^ 4) / (12 * b1 ^ 2)

)

}

g <- function(x, b0, b1) { exp((-b0 ^ 2 + 2 * b0 * x - x ^ 2) / (2 * b1))

}

G <- function(down, up, b0, b1) { integrate(function(x) g(x, b0, b1), down, up)$value

}

g1 <- function(i, down, up, b0, b1) { integrate(function(x) F1(i, x, b0, b1) * g(x, b0, b1),
```

```
down, up)$value }

F2 <- function(i, j, x, b0, b1) { min = min(i, j) max = i + j - min switch(min+1,

switch(max+1,

1 / b1,

                    –  (x - b0) / (b1 ^ 2),

(b0 ^ 2 - x ^ 2) / (3 * b1 ^ 2),

(b0 ^ 3 - b0 * x ^ 3) / (3 * b1 ^ 2)

),

switch(max+1,

NULL,

                    –  (x - b0) / (4 * b1 ^ 3),

                    –  (2 * x ^ 3 - 3 * b0 * x ^ 2 + b0 ^ 3) / (12 * b1 ^ 3),

                    –  (3 * x ^ 4 - 4 * b0 * x ^ 3 + b0 ^ 4) / (24 * b1 ^ 3)

),

switch(max + 1,

NULL,

NULL,

                    –  (3 * x ^ 4 - 4 * b0 * x + b0 ^ 4) / (24 * b1 ^ 3),

                    –  (2 * x ^ 5 - 3 * b0 * x ^ 4 + b0 ^ 3 * x ^ 2) / (3 * b1 ^ 3)

),

                    –  (5 * x ^ 6 - 6 * b0 * x ^ 5 + b0 ^ 6) / (15 * b1 ^ 3)

)

}

G2 <- function(i, j, down, up, b0, b1) { integrate(function(x)(F1(i, x, b0, b1) * F1(j, x, b0, b1) + F2(i, j,

x, b0, b1)) * g(x, b0, b1), down, up)$value

}

MCSim_trunor <- function(Fun, down, up, b0, b1) { X <- rtmvnorm(100000, b0, sqrt(b1), down, up)

lapply(X, Fun) return(mean(x))
```

```
}

Exp_F2 <- function(i, j, down, up, b0, b1) {

MCSim_trunor(function(x) F2(i, j, x, b0, b1), down, up, b0, b1)

}

D = c()

I = Matrix(nrow=4, ncol=4) for (i in seq(0, 3)) {

D[i + 1] <- sum(apply(as.matrix(Data) ,2, function(x) F1(i, x, b0, b1))) - N * g1(i, down, up, b0, b1) / G(down, up, b0, b1) for (j in seq(0, i)) {

print(paste("i",i,",j",j))

I[i+1, j+1] = N * (Exp_F2(i, j, down, up, b0, b1) - (G2(i, j, down, up, b0, b1) * G(down, up, b0, b1) - g1(j, down, up, b0, b1) * g1(i, down, up, b0, b1)) / G(down, up, b0, b1) ^ 2) if (i != j) { I[j+1, i+1] = I[i+1, j+1] }

}

}

LMstat <- t(D)%*%inv(as.matrix(I))%*%D

test_res <- c(LMstat,qchisq(.99, df=2),qchisq(.95, df=2),qchisq(.90,

df=2)) names(test_res) <- c("LM statistic", "99%", "95%", "90%") return(test_res)

}
```

2. Assessment of MLE of Truncated Normal Distribution

2.1 Simulation Methodology

The simulation object is to assess the MLE estimator (distribution, bias, etc...). The algorithm detailed below made 100,000 iterations of estimating the normal parameters of 10, 100 and 1000 (-1,1)truncated standard normal variables. As there is no computational formula to calculate the MLE estimators directly, we are in fact assessing the practicality of the use of the MLE method and not its theoretical qualities.

The baseline is the case of $\mathcal{N}_{-1}^{1}(0,1)$, then we change each time the truncation range in order to study its impact on the MLE performance

2.2 Calculation Script (R Language)

Algorithm 2

```
install.packages("tmvtnorm") library("tmvtnorm")

#set the distribution parameters

lower = d upper = u mu = m sigma = s
```

```
#set the simulation parameters and variables repetion = 100000

mu_mle = sigma_mle = matrix(nrow = 1, ncol = 3)

for (i in seq(1,repetion)) {

for (size in seq(1:3)) {

#Create a sample of (lower,upper)-truncated normal(mu,sigma) X <- rtmvnorm(n=10^size, mu, sigma, lower, upper) method <- "BFGS"

#Estimate the parameters of the created sample using MLE mle.fit1 <- mle.tmvnorm(X, lower=lower, upper=upper) #Extract the estimated variables for each sample size

mu_mle[size] <- mle.fit1@coef[1] sigma_mle[size] <- mle.fit1@coef[2]

}

#Append the estimated variables to the corresponding csv file for further treatments write.table(mu_mle, file="mu_mle.csv", append = TRUE, col.names = FALSE, sep = ",")

write.table(sigma_mle, file="sigma_mle.csv", append = TRUE, col.names = FALSE, sep = ",")

}
```

Copyright Disclaimer

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/)