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The Evolution of Firm Connections in Networks: A Differential Game Approach

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Abstract

Optimal control theory can be employed to gain novel insights on the self-organization and structure of networks. We develop a Cournot differential game to analyse the evolutionary dynamics of firm connections within a network. We determine the feedback (Markovian) Nash equilibrium strategies and the steady state of the model and identify the key factors affecting the strategic choice of network firms. Our model confirms the empirical evidence that network firms tend to increase their own connections over time; moreover, such growing connections are mainly affected by the market size and the spillover rate.

Keywords: firm network, differential game, oligopoly

1. Introduction

Network agreements among firms refer to collaborative relationships and partnerships between different companies, mostly small-medium enterprises. Networks are important for several reasons. First, firms can share resources, knowhow, and capabilities (Bullinger et al., 2004); second, cooperation fosters product and process innovation allowing firms to become more competitive (Teirlinck and Spithoven, 2013); third, partnership can provide access to larger international markets (Ojala, 2009) and increase bargaining power and reputation (Lechner et al., 2006); finally, network agreements allow firm to share risks, mitigating uncertainty by reducing production and transaction costs (Croom, 2001). The strength of business networks is that enterprises, while cooperating with each other, maintain their complete autonomy and specialization; moreover, being a member of a network generates significant improvements in economic performance of firms such as total turnover and added value (Cisi et al., 2020).

Given the positive externalities generated by the participation in a network, we should observe both an increase in the number of networks and a growth in their average size over time. Tomasiello et al. (2016) studied the evolution of R&D networks for different manufacturing sectors over a 24-year period distinguishing two contrasting phases: a rise phase, where networks grow in number and density (number of links), and a fall phase, where alliances are more fragmentated. However, in both phases, network density is higher in high-tech industries such as Computer Software, Electronic Components and Pharmaceutical, characterised by high levels of spillovers. Carbonara (2002) analyses the evolution of Italian Industrial Districts, showing that the formation of structured inter-firm networks is mainly driven by the strategies of growth adopted by the leader firms. Hite and Hesterly (2001) propose that the networks evolve according to firm's resource challenges and needs. In the beginning, networks are characterised by socially embedded ties while, in the final stage of their evolution, ties are based on a calculation of economic costs and benefits.

Further evidence comes from Italy. Figure 1 reports the average number of links formed in Italy by each network firm over the 2010-2023 period.[1] It emerges that network firms have doubled their own partners within ten years.

From a theoretical perspective, several papers have described the process of business network formation, providing important insights about the key factors affecting such process. The seminal paper by Goyal-Moraga Gonzalez (2001) studied the stability of R&D networks among Cournot oligopolists, showing that firms tend to form increasingly dense networks even at the expense of aggregate profits. Opposite results were obtained by Goyal and Joshi (2003) which extended the analysis to Bertrand competition. They showed that in a context characterised by strong price competition, firms tend to form small stable networks arising a trade-off between stability and efficiency in networks.

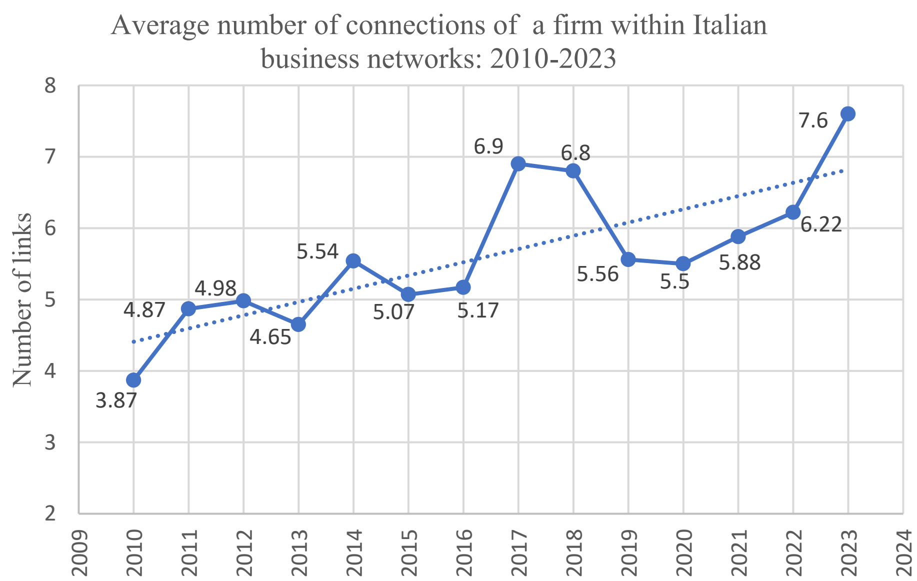

Figure 1. The Average Number of Links Formed by One Firm within a Network, Increased from 2010 to 2023. *Source*: Our Elaboration on InfoCamere Data.[2]

Correani et al. (2014) showed that the number of firms participating in a network is strongly affected by market size, spillover rate (absorptive capacity) and the number of potential partners. More recently, Di Dio and Correani (2017, 2020), proposed theoretical models to examine network formation in Hotelling-type oligopolies. They found that networks improving product quality are generally denser than those driven by process innovation. Moreover, the number of network firms increases when firms feature low vertical differentiation but high horizontal differentiation. Also in these models, the spillover rate plays a key role in determining the structure of the network.

All the above-mentioned contributions share the common feature of considering static games which, however, does not allow to study the growth process of the number of connections within a business network over time. Our paper contributes to this strand of the literature by adopting a dynamic approach. We consider a Cournot oligopoly where each firm can decide to cooperate with the others forming pair-wise collaboration links to reduce its own marginal cost. We suppose such positive externality to evolve according to a dynamic process depending on the number of connections of each single firm, the rate of spillover, and the rate of decay. Therefore, our model is structured as a differential game where each firm's strategies (output and number of connections) evolve over time, possibly converging to a stable equilibrium. We replicate the growing trend of firm connections within a network which is observed in the real world and identify the key factors affecting the dynamic path. First, we identify the feedback (Markovian) Nash equilibrium strategies which converge to a unique steady state; second, we show that each firm's connections increase with market size and spillover rate.

The rest of the paper is organised as follows. Section 2 presents the model and discuss the main results. Section 3 concludes. In the appendix we report the proof of the main results.

2. The Model

Consider an oligopoly market over an infinite (continuous) time horizon, $t \in [0, \infty)$. Firms are Cournot competitors and supply a homogeneous good, whose market demand function is $p_t = a - bQ_t$ at any time $t \in [0, \infty)$, with N denoting the number of firms, $a > 0$, $b > 0$ and $Q_t = \sum_{j=1}^{N} q_{it}$ being the sum of all firms' output levels, $i \in \{1,2,3, \ldots, N\}$. Production takes place at a constant returns to scale, with a constant marginal cost $c \in (0, a)$, common to all N firms. Moreover, each firm i can form alliances (pair-wise collaboration links) with other firms, in order to reduce its own marginal cost. Thus, firm i's instantaneous cost function is $C_i = (c - x_{it})q_{it}$ where x_{it} indicates the efficiency (cost reduction) firm i obtains from the collaborative links. The level of cost reduction x_{it} evolves over time according to the following dynamics:

$$\frac{dx_{it}}{dt} \equiv \dot{x}_{it} = \alpha m_{it} - \delta x_{it}, \quad x_{i0} = 0 \tag{1}$$

where $0 \leq m_{it} \leq N - 1$ is the number of links formed by firm i, $\alpha > 0$ is the spillover rate and $\delta > 0$ is the decay rate of x_{it}. According to equation (1), marginal costs are declining in the number of links (see Goyal and Joshi, 2003). Forming a network of alliances is costly and we assume a quadratic cost function given by $C_{mi} = \gamma_m \frac{m_{it}^2}{2}$ with $\gamma_m > 0$.

At this point, firm i's instantaneous profit function writes as follows:

$$\pi_{it} = \left[a - b\sum_{j=1}^{N} q_{it}\right]q_{it} - (c - x_{it})q_{it} - \gamma_m \frac{m_{it}^2}{2} \, . \tag{2}$$

Firm i chooses q_{it} and to maximise the discounted individual profit flow:

$$J_{it} = \int_0^{\infty} \pi_{it} e^{-\rho t}\, dt, \tag{3}$$

s.t. the state dynamics (1). Parameter $\rho > 0$ represents the constant discount factor common to all firms in the industry. A feedback solution of the problem, which allows the firms to choose their quantities q_{it} and number of cooperative links m_{it} contingent upon the state of the game, has to satisfy the following Bellman equation:

$$\rho V_{it} = \max_{q_{it}, m_{it}} \left\{\pi_{it} + \frac{dV_{it}}{dx_{it}} \dot{x}_{it}\right\}, \tag{4}$$

where we set $V_{it} = A + Bx_{it} + Dx_{it}^2$. Solving the problem and imposing symmetry across quantities and number of links, i.e. $q_{it} = q_{jt} = q_t$ and $m_{it} = m_{jt} = m_t$ for all $i \neq j$, we obtain the following result:

Proposition 1: *The symmetric feedback (Markovian) Nash equilibrium strategies of the game are:*

$$q_t^* = \frac{a-c}{b(N+1)} + \frac{1}{b(N+1)} x_t \qquad \text{and} \qquad m_t^* = \frac{\alpha}{\gamma_m}(B + 2Dx_t) \tag{5}$$

$$\text{where} \quad D = \frac{(2\delta+\rho) \pm \sqrt{(2\delta+\rho)^2 - \frac{8\alpha^2}{\gamma_m b(N+1)^2}}}{\frac{4\alpha^2}{\gamma_m}} \quad \text{and} \quad B = -\frac{2(a-c)}{b(N+1)^2} \times \frac{1}{\frac{2\alpha^2}{\gamma_m}D - \delta - \rho} \, .$$

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Proof: see the appendix

Differentiating (5) with respect to time and using the dynamic constraint (1) we obtain a system consisting of three linear differential equations. It is the main tool for the analysis of optimal solutions of the model:

$$\dot{q}_t = \frac{1}{b(N+1)}\dot{x}_t \ , \tag{6}$$

$$\dot{m}_t = \frac{2\alpha D}{\gamma_m}\dot{x}_t \ , \tag{7}$$

$$\dot{x}_t = \alpha m_t - \delta x_t \ . \tag{8}$$

According to the system (6-8), both trajectories of q_t^* and m_t^* will converge to a steady state if the state trajectory x_t^* converges. Conditions for convergence are stated in the following proposition:

Proposition 2: *If* $\gamma_m > \frac{2\alpha^2}{\delta(\delta+\rho)b(N+1)}$ *, then the feedback (Markovian) Nash equilibrium strategies (5) will converge to the unique steady state* (q_∞, m_∞) *with* $q_\infty = \frac{a-c}{b(N+1)} + \frac{1}{b(N+1)}\frac{\alpha}{\delta}m_\infty$, $x_\infty = \frac{\alpha}{\delta}m_\infty$ and $m_\infty = \frac{\alpha\delta B}{\gamma_m\delta - 2\alpha^2 D}$.

Proof: see the Appendix

A phase diagram depicting the relationship between x_t and m_t is provided in figure 2. The relevant phase space is the set $\{(x_t, m_t) | x_t \geq 0, \ 0 \leq m_t \leq N-1\}$. The solid line is $x = \frac{\alpha}{\delta}m$, which is the locus of all points at which $\dot{x} = \dot{m} = \dot{q}$. It is called the $(\dot{x} = 0)$-isocline. The isocline divides the phase space into two regions each of which is characterized by a unique direction of the flow determined by (7) and (8). These direction are indicated in the figure by arrows. Given that $\dot{m} = 0$ only if $\dot{x} = 0$, the isocline is the set of all the fixed points (steady states) of the system. For any admissible set of parameters $(a, b, c, \alpha, \delta, \gamma_m, N, \rho)$ and the initial state $x_0 = 0$ there exists a unique solution of the system (6-8) which converges to one of the steady states on the isocline. In other words, for a given set of parameters, the time path $\{m_t\}_{t=0}^{\infty}$ will start at a value $m_0 = \frac{\alpha}{\gamma m}B$ which is greater than 0 and less than $N-1$. As shown in the phase diagram, the value of m_t will increase over time and converge to m_∞ as $t \to \infty$. As shown in the phase diagram, trajectory is a straight path according to the relationship $m_t^* = \frac{\alpha}{\gamma m}(B + 2Dx_t)$.

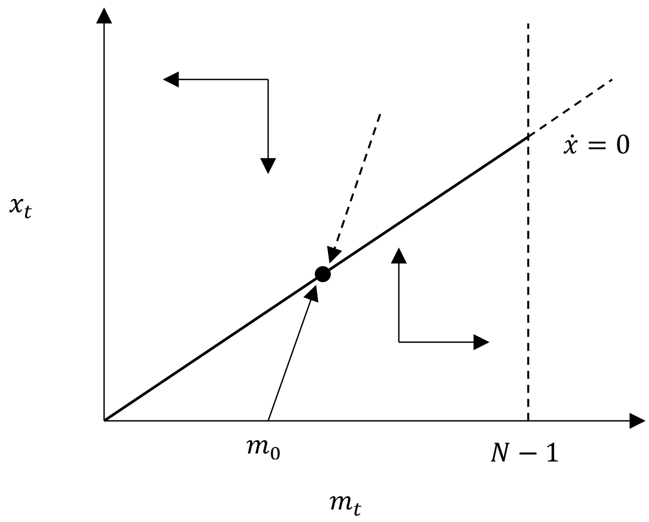

Figure 2. Phase Diagram – the Number of Connections of a Firm Increases over Time, Converging to the Steady State

Substituting the optimal number of links in (5) into the system dynamics we obtain the linear, autonomous initial value problem for the state trajectory,

$$\dot{x}_t = \frac{2\alpha^2}{\gamma_m} B + \left(\frac{2\alpha^2}{\gamma_m} D - \delta\right) x_t,\ x_0 = 0, \tag{9}$$

which can be solved explicitly. This yields:

$$x_t = \frac{\alpha^2 B}{\gamma_m \delta - 2\alpha^2 D}\left[1 - e^{\left(\frac{2\alpha^2 D}{\gamma_m} - \delta\right)t}\right]. \tag{10}$$

Equation (10) confirms that the state variable x is increasing over time and converging to the positive steady state value $x_\infty = \frac{\alpha^2 B}{\gamma_m \delta - 2\alpha^2 D}$.

Observe that it must be $0 \leq m_\infty \leq N - 1$. Nonnegativity is guaranteed by the stability condition $D < \frac{\delta \gamma_m}{2\alpha^2}$ discussed in the appendix, whereas m_∞ is no larger than $N - 1$ only for sufficiently low levels of market size $a - c$, precisely

$$m_\infty \leq N - 1 \text{ if } \quad a - c \leq \frac{b\gamma_m (N-1)(N+1)^2}{4\alpha\delta}\left(-\rho + \sqrt{(2\delta + \rho)^2 - \frac{8\alpha^2}{\gamma_m b (N+1)^2}}\right) \equiv \bar{a}. \tag{11}$$

Condition (11) leads to following result:

Proposition 3: *Let us assume that condition (11) holds then, the higher the market size the denser the network that oligopolistic firms will form. In the long run equilibrium, firms will form a complete network only if* $a - c = \bar{a}$.

Proof: Differentiating m_∞ with respect to $a - c$ we obtain that $\frac{\partial m_\infty}{\partial (a-c)} \geq 0$. From condition

(11) It follows that m_{∞}^{*} increases with the market size, converging to the complete network when $a - c = \bar{a}$.

According to proposition 3, oligopolistic firms tend to form denser networks as the market size converge to its threshold level $\bar{a}$.

It is important to stress that $\bar{a}$ increases with the number of firms N and the cost of links γ_m whereas decreases with the spillover rate α. In other words, for a given market size, we will observe a lower network degree in more competitive markets characterised by a lower level of spillovers and higher costs of link formation. The economic intuition behind this result is that increasing competition lowers the efficiency that each firm can absorb from its partners (precisely, x_{∞} decreases with N), inducing firms to form smaller networks. Obviously, this effect is boosted by decreasing spillover rate and increasing cost of cooperative links.

3. Conclusion

We develop a Cournot-type differential game to analyse the dynamic of firm connections within a network. The state variable is represented by the degree of cost reduction generated by collaboration, and it evolves according to the number of connections and the rate of spillover. Networks tend to become denser over time, and the number of connections of each network firm follows a growing path which converge to a steady state. Larger market size makes collaboration more profitable prompting firms to increase connections. Similar results are obtained with an increasing spillover rate.

Based on our results, we expect denser networks in growing markets and when firms can benefit significantly from sharing information and expertise (high spillovers). Tomasiello et al. (2016) confirm this result for high tech industries such as Pharmaceutical, Computer Software and electronic component.

As a further development of the model one could describe the evolution of the state variable x as a stochastic process of the type $dx_{it} = (\alpha m_{it} - \delta x_{it})dt + \sigma x_{it} dW_{it}$ where W_{it} denotes the realization of the Wiener process at time t, whereas $\sigma > 0$ measures the influence of the Wiener process on the dynamic of the state variable x. According to this formulation, each firm is assumed to take into account the relative volatility that its choice might transmit to the system. The higher is the effort in increasing efficiency by cooperative links, the more uncertainty will enter the system.

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Notes

Note 1. We consider 2010 as the starting date since in that year the Italian Government introduced the law on networks.

Note 2. Provided by the Italian Chamber of Commerce.

Appendix

Proposition 1

The game at hand takes a linear quadratic form, with the following Bellman equation:

$$\rho V_{it} = \max_{q_{it}, m_{it}} \left\{ \left[a - b \sum_{j=1}^{N} q_{it} \right] q_{it} - (c - x_{it}) q_{it} - \gamma_m \frac{m_{it}^2}{2} + \frac{dV_{it}}{dx_{it}} [\alpha m_{it} - \delta x_{it}] \right\}, \quad (20)$$

and therefore, we set $V_{it} = A + Bx_{it} + Dx_{it}^2$ .The first order conditions are :

$$a - 2bq_{it} - b \sum_{j \neq i}^{N} q_{jt} - c - x_{it} = 0 \ , \quad (21)$$

$$-\gamma_m m_{it} + \alpha(B + 2Dx_{it}) = 0 \ , \quad (22)$$

whereby, imposing symmetry, the firm's Markov strategies are

$$q_t^* = \frac{a-c}{b(N+1)} + \frac{1}{b(N+1)} x_t \ , \quad (23)$$

$$m_t^* = \frac{\alpha}{\gamma_m} (B + 2Dx_t). \quad (24)$$

By substituting conditions (23) and (24) into (20) and collecting terms with equal power of x_t we see that, in order for (20) to be satisfied, it must hold that

$$\frac{(a-c)^2}{b(N+1)^2} + \frac{\alpha^2}{2\gamma_m} B^2 - \rho A = 0 \ , \quad (25)$$

$$\frac{2(a-c)}{b(N+1)^2} + \frac{2\alpha^2}{\gamma_m} D^2 - s\delta D - \rho D = 0 \ , \quad (26)$$

$$\frac{1}{b(N+1)^2} + \frac{2\alpha^2}{\gamma_m} BD - B(\delta + \rho) = 0 \ . \quad (27)$$

This yields:

$$A = \left\{ \frac{(a-c)^2}{b(N+1)^2} + \frac{2\alpha^2}{\gamma_m} \left[-\frac{2(a-c)}{b(N+1)^2} \times \frac{1}{\frac{2\alpha^2}{\gamma_m} D - \delta - \rho} \right]^2 \right\} \frac{1}{\rho} , \tag{28}$$

$$B = -\frac{2(a-c)}{b(N+1)^2} \times \frac{1}{\frac{2\alpha^2}{\gamma_m} D - \delta - \rho} , \tag{29}$$

$$D = \frac{(2\delta+\rho) \pm \sqrt{(2\delta+\rho)^2 - \frac{8\alpha^2}{\gamma_m b(N+1)^2}}}{\frac{4\alpha^2}{\gamma_m}} . \tag{30}$$

Proposition 2

Differentiating (23) and (24) w.r.t. time we obtain the following system of linear differential equations:

$$\dot{q}_t = \frac{1}{b(N+1)} \dot{x}_t , \tag{31}$$

$$\dot{m}_t = \frac{2\alpha D}{\gamma_m} \dot{x}_t , \tag{32}$$

$$\dot{x}_t = \alpha m_t - \delta x_t . \tag{33}$$

This system describes the dynamics of both control and state variables of the model which converge to a steady state $(m_\infty, q_\infty, x_\infty)$ only if x_t converges to a fixed point. Thus, let us rewrite equation (33) as

$$\dot{x}_t = \frac{\alpha^2}{\gamma_m} B + \left(\frac{2\alpha^2}{\gamma_m} D - \delta \right) x_t, \tag{34}$$

where we have used $m_t^* = \frac{\alpha}{\gamma_m}(B + 2Dx_t)$.

Observe that the dynamic described by equation (34) converges to a positive fixed point x_∞ only if $D < \frac{\delta\gamma_m}{2\alpha^2} \equiv \bar{D}$ and $B > 0$. It is straightforward to show that the positivity of B is guaranteed by taking the smaller root in (30) i.e. $D = \frac{2\delta+\rho-\sqrt{(2\delta+\rho)^2-\frac{8\alpha^2}{\gamma_m b(N+1)^2}}}{4\alpha^2\gamma_m^{-1}}$ where we assume a positive discriminant. Moreover, it is easy to show that $D < \bar{D}$ if $\gamma_m > \frac{2\alpha^2}{\delta(\delta+\rho)b(N+1)^2}$ which is a reasonable condition for a sufficiently high number of firms.

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No additional data are available.